Math 246A Homework 11

Brendan Connelly

Due Friday, December 5, 2025

Problem 10.9

Let f be a holomorphic function near $z_0 \in \mathbb{C}$ such that $f'(z_0) \neq 0$. Its Schwarzian derivative is defined as

$$\mathscr{S}(f)(z) := \frac{d^2}{dz^2}\log(f'(z)) - \frac{1}{2}\left(\frac{d}{dz}\log(f'(z))\right)^2 \tag{10.5}$$

for z near z_0. **(a)** Find an explicit expression of $\mathscr{S}(f)$ in terms of f', f'', and f'''.

We can directly calculate

$$\begin{aligned}\mathscr{S}(f) &= \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \\ &= \frac{f'''f' - (f'')^2}{(f')^2} - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \\ &= \frac{2f'''f' - 3(f'')^2}{2(f')^2} \\ &= \frac{f'''}{f'} - \frac{3(f'')^2}{2(f')^2}\end{aligned}$$

□

……

(b) Suppose h is holomorphic near $u_0 \in \mathbb{C}$ with $z_0 = h(u_0)$ and $h'(u_0) \neq 0$. Show that the chain rule

$$\mathscr{S}(f \circ h) = \big(\mathscr{S}(f) \circ h\big)(h')^2 + \mathscr{S}(h) \tag{10.6}$$

holds locally near u_0.

In the previous question (10.8), we defined the logarithmic derivative $\mathcal{L}(f) = \dfrac{f'}{f}$ and proved that $\mathcal{L}(fg) = \mathcal{L}(f) + \mathcal{L}(g)$ in a neighborhood where both f, g are holomorphic, and similarly the chain rule $\mathcal{L}(f \circ h) = (\mathcal{L}(f) \circ h)\, h'$. We can leverage both of these properties by rewriting

$$\mathscr{S}(f) = (\mathcal{L}(f'))' - \tfrac{1}{2}\left(\mathcal{L}(f')\right)^2.$$

Then we plug in $f \circ h$:

$$\mathscr{S}(f \circ h) = \big(\mathcal{L}\big((f \circ h)'\big)\big)' - \tfrac{1}{2}\left(\mathcal{L}\big((f \circ h)'\big)\right)^2$$

$$
\begin{aligned}
&= \big(\mathcal{L}\big((f'\circ h)h'\big)\big)' - \tfrac{1}{2}\left(\mathcal{L}\big((f'\circ h)h'\big)\right)^2 \\
&= \big(\mathcal{L}(f'\circ h) + \mathcal{L}(h')\big)' - \tfrac{1}{2}\big(\mathcal{L}(f'\circ h) + \mathcal{L}(h')\big)^2.
\end{aligned}
$$

Set

$$A := \mathcal{L}(f'\circ h), \qquad B := \mathcal{L}(h').$$

Then

$$\mathscr{S}(f\circ h) = (A' + B') - \tfrac{1}{2}(A^2 + 2AB + B^2), \qquad \mathscr{S}(h) = B' - \tfrac{1}{2}B^2$$

Hence

$$\mathscr{S}(f\circ h) - \mathscr{S}(h) = A' - \tfrac{1}{2}A^2 - AB$$

Now we use the chain rule for $\mathcal{L}$ again

$$\mathcal{L}(f'\circ h) = (\mathcal{L}(f')\circ h)\, h'$$

Now define

$$F := \mathcal{L}(f')\circ h$$

so $A = Fh'$. Differentiating,

$$A' = (Fh')' = F'h' + Fh''$$

and by the chain rule,

$$F' = (\mathcal{L}(f')'\circ h)\, h'$$

Thus

$$A' = (\mathcal{L}(f')'\circ h)\,(h')^2 + (\mathcal{L}(f')\circ h)\, h''$$

Also we can say,

$$A^2 = F^2(h')^2 = \big(\mathcal{L}(f')\circ h\big)^2(h')^2 \qquad AB = Fh'\,B = (\mathcal{L}(f')\circ h)\,h'\,\mathcal{L}(h')$$

Substituting into $\mathscr{S}(f\circ h) - \mathscr{S}(h) = A' - \tfrac{1}{2}A^2 - AB$, we get

$$
\begin{aligned}
\mathscr{S}(f\circ h) - \mathscr{S}(h) &= (\mathcal{L}(f')'\circ h)\,(h')^2 + (\mathcal{L}(f')\circ h)\,h'' - \tfrac{1}{2}\big(\mathcal{L}(f')\circ h\big)^2(h')^2 - (\mathcal{L}(f')\circ h)\,h'\,\mathcal{L}(h') \\
&= (h')^2\left[\mathcal{L}(f')'\circ h - \tfrac{1}{2}\big(\mathcal{L}(f')\circ h\big)^2\right] + (\mathcal{L}(f')\circ h)\big(h'' - h'\,\mathcal{L}(h')\big)
\end{aligned}
$$

But $\mathcal{L}(h') = \dfrac{h''}{h'}$, so we obtain

$$\mathscr{S}(f\circ h) - \mathscr{S}(h) = (h')^2\left[\mathcal{L}(f')'\circ h - \tfrac{1}{2}\big(\mathcal{L}(f')\circ h\big)^2\right] = \big(\mathscr{S}(f)\circ h\big)\,(h')^2$$

Therefore,

$$\mathscr{S}(f\circ h) = \big(\mathscr{S}(f)\circ h\big)\,(h')^2 + \mathscr{S}(h)$$

locally near u_0. □

Problem 12.2

Let $S \in$ Möb and $S \neq \mathrm{id}_{\widehat{\mathbb{C}}}$. Show that S is conjugate to a Möbius transformation T of precisely one of the following types:

$$T(z) = \begin{cases} z+1, & \text{(parabolic)}, \\ \lambda z,\ \lambda \in \partial\mathbb{D} \setminus \{1\}, & \text{(elliptic)}, \\ \lambda z,\ \lambda \in (0,\infty) \setminus \{0\}, & \text{(hyperbolic)}, \\ \lambda z,\ \lambda \in \mathbb{C} \setminus (\partial\mathbb{D} \cup \mathbb{R}), & \text{(loxodromic)}. \end{cases}$$

This follows quickly from the matrix representation of Möbius transformations and linear algebra facts. We know that we can represent a Möbius transformation by a 2×2 matrix with entries in $\mathbb{C}$ such that $ad - bc \neq 0$. This determinant condition is equivalent to saying the matrix representation of S, which I will denote by A, does not have zero as an eigenvalue. Möbius transformations compose by multiplying their matrix representations. Hence, the matrix representation for T, which I will denote B, must just be similar to A up to scaling. Since we assume that $S \neq \mathrm{id}_{\widehat{\mathbb{C}}}$, we can say that $A \neq cI_2$, where I_2 denotes the 2×2 identity matrix. Hence B cannot be either. Then, we can consider two cases. First, suppose A is diagonalizable. Then,

$$B = VAV^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where V is the corresponding change of coordinates matrix that represents the transformation U. This produces the Möbius transformation T where we define $\lambda = \frac{\lambda_1}{\lambda_2}$.

$$z \mapsto \frac{\lambda_1 z}{\lambda_2} = \frac{\lambda_1}{\lambda_2} z = \lambda z$$

We can observe that λ must be non-zero (and well defined) as no eigenvalue can be zero. We can also observe that $\lambda_1 \neq \lambda_2$ as otherwise we are in the case $B = cI_2$. Then, we can deduce that $\lambda \neq 0, 1$. Hence, we fall exactly into one of the last three cases.

In the other case, λ is an eigenvalue with multiplicity two and geometric multiplicity one. Hence, A is similar to the Jordan block

$$B = VAV^{-1} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

which exactly gives that $T(z) = z + \frac{1}{\lambda}$. Now, we need to show that $z \to z + c$ is conjugate to $z \to z + 1$ for $c \in \mathbb{C}$ arbitrary. We immediately have this from the fact about Jordan blocks that we can scale the value of 1 by any non-zero scalar by symmetry. We can also explicitly see this via the change of basis matrix $D = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix}$ up to multiplication by scalars which preserves the transformation. This produces

$$\begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Hence, we have that A is similar to the above matrix (via the composition of matrices DV) which corresponds precisely to the Möbius transformation $z \to z + 1$.

So far, we have proven that all Möbius transformations must lie in at least one of the above cases. We know from the linear algebra we have used thus far that parabolic case is mutually exclusive from the other three cases–as we cannot have a matrix that is not diagonalizable similar to matrix that is. Hence, to complete the proof, it suffices to check when $S(z) = \lambda z$ and $T(z) = \mu z$ are conjugate for $\lambda, \mu \in \mathbb{C}$. This follows from the transitivity of conjugation, which is immediate from the definition as Möbius transformations are biholomorphisms.

We can do this by analyzing the fixed points of a Möbius transformation. Both S, T must have exactly $\{0, \infty\}$ as fixed points. By question 12.1, we know that Möbius transformation preserve fixed points. Suppose

U is the Möbius transformation that is the conjugation map. Then, $U(0) \neq U(\infty) \in \{0, \infty\}$. If we suppose that $0 \mapsto 0, \infty \to \infty$, we plug into our definition and see our Möbius transformation U must be in the form $z \mapsto az$. In the alternate case, we suppose that $0 \mapsto \infty, \infty \to 0$, our Möbius transformation U must be in the form $z \mapsto \frac{a}{z}$. Then, we see

$$T = U \circ S \circ U^{-1} \qquad T(z) = a(\lambda(\frac{1}{a}z)) \quad \text{or} \quad T(z) = \frac{a}{\lambda(\frac{a}{z})} = \frac{1}{\lambda}z$$

Hence, $\mu z = \lambda z$ or $\mu z = \frac{1}{\lambda}z$. Thus, we have that T, S are conjugate if and only if $T = S$ or $\mu = \frac{1}{\lambda}$. Thus, if $\lambda \in \partial\mathbb{D} \setminus \{1\}$, S can only be conjugate to another T with λ in the region by reflecting across the real axis, which stays in $\partial\mathbb{D} \setminus \{1\}$. Negative one will be mapped to itself. Taking a reciprocals is an involution so we need not worry about more than what λ is immediately mapped to. If $\lambda \in (0, \infty) \setminus \{1\}$, taking reciprocals is again closed in $(0, \infty) \setminus \{1\}$. The same is then true with $\lambda \in \mathbb{C} \setminus (\partial\mathbb{D} \cup \mathbb{R})$. Hence, each of the last three cases are disjoint. So, we have that each S must be conjugate to exactly one of the above cases. Lastly, it's worth noting that proving all four cases in one direction logically gives all four if and only if statements since every transformation needs to fall into exactly one category. □

Problem 12.4

Let

$$S(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1.$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the coefficient matrix of S, and let $\operatorname{tr}(A) = a + d$ be its trace. Show that S is

$$\begin{cases} \text{parabolic} & \text{iff } \operatorname{tr}(A)^2 = 4, \\ \text{elliptic} & \text{iff } \operatorname{tr}(A)^2 \in [0, 4), \\ \text{hyperbolic} & \text{iff } \operatorname{tr}(A)^2 \in (4, +\infty), \\ \text{loxodromic} & \text{iff } \operatorname{tr}(A)^2 \notin [0, +\infty). \end{cases}$$

These types correspond to the classification in Problem 12.2.

We can continue analyzing via linear algebra facts as above. First, we note that similar matrices have the same trace. In 12.2, we showed that S is parabolic if and only if its matrix representation is similar to a matrix of the form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This does in fact have determinant one so we can calculate its trace squared which is, $(1 + 1)^2 = 4$. For the following cases, we know that the matrix representation is conjugate to

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$$

We can scale this as in question 12.3 to make the determinant one, giving us

$$\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$$

where the square root symbol denotes the principal branch of the square root. If λ is on the negative real line, we can set $\lambda = -\lambda$. In any case, we can then take the trace squared of this matrix, producing

$$\mathrm{tr}(A)^2 = \lambda + \frac{1}{\lambda} + 2$$

In the elliptic case, we have $\lambda \in \partial\mathbb{D} \setminus \{1\}$. Here, $\lambda + \frac{1}{\lambda} = 2\mathrm{Re}(\lambda) \in [-2, 2)$. Hence, $\mathrm{tr}(A)^2 \in [0, 4)$.

In the hyperbolic case, we have $\lambda \in (0, \infty) \setminus \{1\}$. By the AM-GM inequality, $\lambda + \frac{1}{\lambda} \geq 2\sqrt{\frac{\lambda}{\lambda}} = 2$. Because $\lambda \neq \frac{1}{\lambda}$ since $\lambda \neq 1$, the inequality is strict. Hence, $\mathrm{tr}(A)^2 \in (4, \infty)$. We know it is not infinity since $\lambda \neq 0, \infty$.

In the loxodromic case, we have $\lambda \in \mathbb{C} \setminus (\partial\mathbb{D} \cup \mathbb{R})$. In polar coordinates, we have $\lambda = re^{i\alpha}$, with $r \neq 1$ and $\alpha \neq \pi k$ for $k \in \mathbb{Z}$. Then, WLOG, we suppose $r > 1$ as the following argument holds with r and $\frac{1}{r}$ flipped.

$$\lambda + \frac{1}{\lambda} + 2 = re^{i\alpha} + \frac{1}{r}e^{-i\alpha} + 2 = (r - \frac{1}{r})e^{i\alpha} + 2\mathrm{Re}\left(\frac{1}{r}e^{-i\alpha}\right) + 2$$

We can see that we will have some real part of the trace (which may cancel to zero) and then a part $(r - \frac{1}{r})e^{i\alpha}$ which must not be in $\mathbb{R}$ since $r - \frac{1}{r} \neq 0$ and $\alpha \neq \pi k$ by assumption. Hence, $\mathrm{tr}(A)^2 \notin [0, \infty)$. □

Problem 12.6

Let C_1 and C_2 be two disjoint circles in $\widehat{\mathbb{C}}$.
(a) Show that there exists a unique pair of distinct points $z, z^* \in \widehat{\mathbb{C}}$ such that z and z^* are symmetric with respect to both circles C_1 and C_2. Hint: Reduce to the case in which one circle is the unit circle $\partial\mathbb{D}$ and the other lies inside $\mathbb{D}$ and is centered on $[0, 1)$.

We can reduce to the case where one circle is mapped to the real line and the other is mapped to a circle in $\mathbb{C}$. If one circle is already the real line, we can just take the identity map. Then, WLOG, let $U(C_1) = \hat{\mathbb{R}}$. Then, let $\tilde{C} = U(C_2)$ which is centered at a with radius R. By the symmetry principle, we can consider this reduced case where one circle is the real line. By the discussion on the geometric description of the circle, points that are symmetric with respect to both circles must satisfy

$$z^* = \overline{z} \qquad z^* = a + \frac{R^2}{\overline{z} - \overline{a}}$$

We can substitute and then say that any pair (z, z^*) must satisfy

$$(z^* - a)(z^* - \overline{a}) = R^2$$

This is a quadratic in z^* which will have two roots with multiplicity. We can say these roots must be distinct as if they are not, then $|z - a| = R$ and $z \in \mathbb{R}$. Hence, a point symmetric with itself would have to lie on both $\hat{\mathbb{R}}$ and $\tilde{C}$, but we know our original circles are disjoint and the disjointness was preserved via a biholomorphism U.

Now, for each root z_i^*, we will have that the pair (z_i, z_i^*) is symmetric with respect to both circles. Thus, we can say the set of all points which are symmetric is exactly the unique set $\{z_1^*, z_2^*\}$. Finally, as U is a conformal map, we have that our distinct unique symmetric points are $\{U^{-1}(z_1^*), U^{-1}(z_2^*)\}$. □

……

(b) Show that there exists $S \in$ Möb such that $S(C_1)$ and $S(C_2)$ are circles in $\mathbb{C}$ centered at 0.

We know there is a unique pair of points z_1, z_2 symmetric to both circles. Let's define the Möbius transformation S such that $S(z_1) = 0$ and $S(z_2) = \infty$. Since a Möbius transformation is triply transitive, such a transformation will exist. I claim that $S(C_1)$ and $S(C_2)$ are circles in $\mathbb{C}$ centered at 0. Let's prove this.

We know then that for both circles, $0, \infty$ are symmetric since symmetry is preserved by Möbius transformations. We also know both circles are not equal to $\hat{\mathbb{R}}$ since $0 \neq \overline{\infty}$ which would be the requirement for those two points to be symmetric on the circle $\hat{\mathbb{R}}$. Furthermore, considering the following expressions with appropriate limits when we have $z = 0, \infty$, we can say that reflection across either circle must satisfy the following where a_i, R_i are the center and radii of each distinct circle.

$$z \mapsto a_i + \frac{R_i^2}{\overline{z} - \overline{a}_i}$$

But, since $0 \mapsto \infty$ and $\infty \to 0$, we have that $a_i = 0$ for both circles. Hence, both are $S(C_1)$ and $S(C_2)$ are circles in $\mathbb{C}$ centered at 0. Thus, exists $S \in$ Möb such that $S(C_1)$ and $S(C_2)$ are circles in $\mathbb{C}$ centered at 0 where S is such that $S(z_1) = 0$ and $S(z_2) = \infty$. $\square$