Math 246A Homework 3

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Problem 2.7

Let (X, d) be a metric space. Recall that $M \subseteq X$ is called connected if the following condition is true: if $U, V \subseteq X$ are open, $M \subseteq U \cup V$, and $U \cap V = \varnothing$, then $M \subseteq U$ or $M \subseteq V$.
Show that $M \subseteq X$ is connected if and only if $\varnothing$ and M are the only subsets of M that are both relatively open and relatively closed.
Note: This statement would be very easy to prove if we replaced the condition $U \cap V = \varnothing$ with $U \cap V \cap M = \varnothing$ in the definition of connectedness.

($\Rightarrow$) To prove the forward direction, suppose for contradiction that there exists $A \subset M$ such that $A, M \setminus A$ are relatively clopen and non-empty. Let $B = M \setminus A$. Then, by the definition of relatively clopen sets, there exists A', B' open in X such that $A = A' \cap M, B = B' \cap M$. Let's define

$$d_A(x) = \inf\{d(a, x) : a \in A\} \quad d_B(x) = \inf\{d(x, b) : b \in B\}$$

For each $x \in A$ and $y \in B$, construct $B(x, \frac{1}{3}d_B(x))$ and $B(y, \frac{1}{3}d_A(y))$. Each distance is strictly positive since both A, B are closed in M. Then, define

$$O_x = A' \cap B(x, \frac{1}{3}d_B(x)) \quad O_y = B' \cap B(y, \frac{1}{3}d_A(y))$$

$$U = \bigcup_{x \in A} O_x \quad V = \bigcup_{y \in B} O_y$$

By construction, U, V are open in X and but neither $M \subset U$ or $M \subset V$. We claim $U \cap V = \varnothing$. Suppose not. Then, take $z \in U \cap V$. Then there are $x \in A$, $y \in B$ with

$$d(z, x) \ < \ \frac{1}{3}\, d_B(x) \leq \frac{1}{3}\, d(x, y), \qquad d(z, y) \ < \ \frac{1}{3}\, d_A(y) \leq \frac{1}{3}\, d(x, y),$$

where the final inequalities use $d_B(x) \leq d(x, y)$ and $d_A(y) \leq d(x, y)$ (since the infimum over B or A is no larger than the distance to the specific point y or x). By the triangle inequality,

$$d(x, y) \leq d(x, z) + d(z, y) < \tfrac{1}{3}\, d(x, y) + \tfrac{1}{3}\, d(x, y) = \tfrac{2}{3}\, d(x, y),$$

which is a contradiction. Hence $U \cap V = \varnothing$.

This proves U, V disconnect the set, a larger contradiction, completing this direction of the proof.

($\Leftarrow$) We need to show the reverse direction. Suppose that $\varnothing, M$ are the only relatively clopen subsets of M. But, if $U, V \subset X$ open with $M \subset U \cup V$, and $U \cap V$ is empty but $M \not\subset U$ and $M \not\subset V$, then we have that $U' = U \cap M$ and $V' = V \cap M$ are both relatively open. Since $U' = M \setminus V'$ it is also relatively closed. Hence, U' is relatively clopen but not empty nor equal to all of M. Hence, we have a contradiction. This completes the proof. □

Problem 2.10

Let M be a non-empty subset of $\mathbb{C}$. A (connected) component C of M is a maximal (with respect to inclusion) connected subset of M. Show that: **(a)** Each component C of M is relatively closed in M.

The closure of a connected set is connected by problem 2.8. Hence, if C is a component of M, and C is not closed in M, we can say that there exists some convergent sequence such that $(x_n) \subset C$ but $x_n \to x \notin C$. Then, $\overline{C}$ (where this is the closure in M) is connected and strictly larger than C since $x \in \overline{C}$. This is a contradiction to being a maximal connected subset. □

(b) If C and C' are distinct components of M, then $C \cap C' = \varnothing$.

We want to show that in general, if $D = A \cap B \neq \varnothing$, for A, B connected subsets of some ambient space X, $A \cup B$ is a connected subset of the ambient space. This is also true by problem 2.9. Suppose $A \cup B$ is disconnected for contradiction. Then, $A \subset U$ or $A \subset V$. If not, $A \cap U$ and $A \cap V$ are disjoint opens in the subspace topology of A, which shows A is disconnected, a contradiction. Hence, WLOG, $A \subset U$ and $B \subset V$. But, $\varnothing \neq A \cap B \subset U \cap V$, which is a contradiction.

Now applying the more general proof, we can say $C \cup C'$ is a connected subset of M strictly larger than either C or C', contradicting the maximality of a component. Therefore, if $C \neq C'$, it must be that

$$C \cap C' = \varnothing.$$

□

(c) Every non-empty connected subset A of M lies in a unique component C of M.

By definition, a component that contains a connected set or a point is the union of all connected sets that contain it. Hence, it lies in a component. However, if it lies in two distinct components, C and C', we have that $C \cap C' = \varnothing$ by (b). Then, $A \not\subset C \cap C'$. Thus, we also have uniqueness. □

(d) M is the (disjoint) union of its components.

Every point $x \in M$ is connected and contained in a unique component C_x of M. Hence,

$$M = \bigcup_{x \in M} C_x$$

We also know that for $x_1, x_2 \in M, C_{x_1} = C_{x_2}$ or $C_{x_1} \cap C_{x_2} = \varnothing$. Hence, this is a disjoint union.

(e) There exists an equivalence relation $\sim$ on M whose equivalence classes are the components of M. Give a concise description of $\sim$ that does not use the concept of a component!

$x \sim y$ for $x, y \in M$ if there exists a connected set containing both x and y. By the four facts above we just showed, this exactly produces equivalence classes that are the connected components. As for proving this is an equivalence relation, reflexivity and symmetry are immediate given that singletons are connected and the definition is symmetric. As for transitivity, if $x \sim y, y \sim z$, there exists connected sets A, B such that $x, y \in A$ and $y, z \in B$. Then, $A \cap B \neq \varnothing$. This implies $A \cup B$ is connected by (b). Hence, $x, z \in A \cup B$ so $x \sim z$. As for the equivalence classes,

$$[x] = \bigcup \{ U \subset M \;\mid\; x \in U \text{ and } U \text{ is connected} \}$$

This is exactly the definition of one connected component. Varying $x \in M$ produces all connected components.

□

Problem 3.1

Let $D = \overline{B(a, r)}$ be a closed Euclidean disk in $\mathbb{C}$ of radius $r > 0$ centered at $a \in \mathbb{C}$.
(a) Suppose $z_1, \ldots, z_n$ are points in D, and $\lambda_1, \ldots, \lambda_n$ are real numbers in the interval $[0, 1]$ with $\lambda_1 + \cdots + \lambda_n = 1$. Show that $\lambda_1 z_1 + \cdots + \lambda_n z_n \in D$.

We want to show that

$$|(\lambda_1 z_1 + \ldots + \lambda_n z_n) - a| \leq r$$

Now consider

$$|(\lambda_1 z_1 + \ldots + \lambda_n z_n) - a| = \left| \sum_{i=1}^{n} \lambda_i z_i - \lambda_i a \right|$$

since $a = \sum_{i=1}^{n} \lambda_i a$. Then, we can say by the triangle inequality, this is

$$\leq \sum_{i=1}^{n} \lambda_i |z_i - a| \;\leq\; \sum_{i=1}^{n} \lambda_i r \;=\; r \sum_{i=1}^{n} \lambda_i \;=\; r$$

□

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(b) Suppose $z \mapsto P(z) = \prod_{k=1}^{n} (z - z_k)$ is a complex polynomial whose zeros $z_1, \ldots, z_n$ lie in D. Show that the zeros of P' also lie in D.

By the product rule, we have that

$$P'(z) = \sum_{k=1}^{n} \left(\prod_{\substack{j=1 \\ j \neq k}}^{n} (z - z_j) \right)$$

We can also rewrite this as

$$P'(z) \;=\; \sum_{k=1}^{n} \frac{P(z)}{z - z_k} \;=\; P(z) \sum_{k=1}^{n} \frac{1}{z - z_k}$$

Let y is a root of $P'(z)$. If y is a root of $P(z)$, we are done. Otherwise, we can say that

$$\sum_{k=1}^{n} \frac{1}{y - z_k} = 0$$

From here, we want to write y as a convex combination of z_k for $k = 1, \ldots, n$. Let's try the following algebraic manipulations:

$$\begin{aligned} 0 &= \sum_{k=1}^{n} \frac{1}{y - z_k} \\ &= \sum_{k=1}^{n} \frac{\overline{y - z_k}}{|y - z_k|^2} \\ &= \sum_{k=1}^{n} \frac{\overline{y} - \overline{z_k}}{|y - z_k|^2} \\ &= \sum_{k=1}^{n} \frac{\overline{y}}{|y - z_k|^2} - \frac{\overline{z_k}}{|y - z_k|^2} \end{aligned}$$

Now, we can rearrange and take the complex conjugate, producing

$$y \sum_{k=1}^{n} \frac{1}{|y - z_k|^2} = \sum_{k=1}^{n} \frac{z_k}{|y - z_k|^2}$$

Then, we have that

$$y = \frac{\sum_{k=1}^{n} \frac{z_k}{|y - z_k|^2}}{\sum_{k=1}^{n} \frac{1}{|y - z_k|^2}} = \sum_{k=1}^{n} \frac{\frac{1}{|y - z_k|^2}}{\sum_{j=1}^{n} \frac{1}{|y - z_j|^2}} z_k$$

We need to show this is a convex combination, with $\lambda_k = \frac{\frac{1}{|y - z_k|^2}}{\sum_{j=1}^{n} \frac{1}{|y - z_j|^2}}$. By convex, we need to show $\sum \lambda_k = 1$. But I claim this is clear. It is particularly clear if we let $r_k = |y - z_k|^2$. Then,

$$\lambda_k = \frac{\frac{1}{r_k}}{\sum_{j=1}^{n} \frac{1}{r_j}} \implies \sum \lambda_k = \frac{\sum_{j=1}^{n} \frac{1}{r_j}}{\sum_{j=1}^{n} \frac{1}{r_j}} = 1$$

Hence, by (a), we are done. □

Problem 3.2

(a) Let $S^1 := \{w \in \mathbb{C} : |w| = 1\}$. Show that the map $f : \mathbb{R} \to S^1$, $t \mapsto f(t) := e^{it}$, is a covering map of $\mathbb{R}$ onto S^1. That is, for each point $w_0 \in S^1$ there exists an open neighborhood $U \subseteq S^1$ of w_0 such that $f^{-1}(U)$ can be written as a countable union

$$f^{-1}(U) = \bigcup_{k \in \mathbb{Z}} U_k$$

of pairwise disjoint open sets $U_k \subseteq \mathbb{R}$ such that $f|_{U_k}$ is a homeomorphism of U_k onto U for each $k \in \mathbb{Z}$.

Let $w_0 = (\cos(\theta), \sin(\theta)) \in S^1$ for $\theta \in [0, 2\pi)$. Then, take an open set

$$U = \{(\cos\phi, \sin\phi) : \phi \in (\theta - \tfrac{\pi}{2},\, \theta + \tfrac{\pi}{2})\} \subset S^1$$

Then,

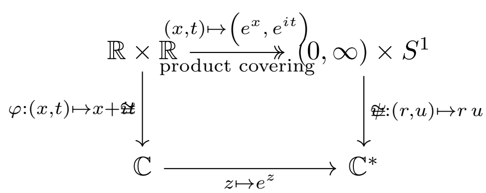
$$f^{-1}(U) = \bigcup_{k\in\mathbb{Z}} \left(\theta - \frac{\pi}{2} + 2\pi k,\, \theta + \frac{\pi}{2} + 2\pi k\right)$$

Now, we can say that $f|_{U_k}$ is a homeomorphism of U_k onto U for each $k \in \mathbb{Z}$. Fix $k \in \mathbb{Z}$. Let's show $f|_{U_k}$ is injective. Suppose $e^{ix_1} = e^{ix_2}$. Then, $e^{ix_1} - e^{ix_2} = 0 \implies x_1 - x_2 \in 2\pi n$ for some n. But since for fixed n, all preimages are bounded by an interval of length 2π; we have that $x_1 - x_2 = 0 \implies x_1 = x_2$. Now, for surjectivity, suppose $y \in U$. Then, $e^{i\theta}$ for a $\theta \in [0, 2\pi)$. Then, we have that $f|_{U_k}^{-1}(\{e^{i\theta}\}) = \theta + 2\pi k$. But, this is in some U_k. Now, we need to show the function $f|_{U_k}^{-1}$ is continuous. Equivalently, we need to show $f|_{U_k}$ takes open sets to open sets since we have already shown it is bijective. By a basic fact from the topology on $\mathbb{R}$, these open sets are the unions of open intervals. We know that $t \to e^{it}$ satisfies this property since sine and cosine do. Alternatively, we can say that for each open set, we can extend the restriction of f to be defined on the closure of its domain since we have sufficiently separated each open interval that composes the preimage $\mathbb{R}$. Hence, we can say that $f|_{\overline{U_k}}$ is a function from a compact space to a hausdorff space, and since it is bijective, we can apply the fundamental theorem of point set topology from Math 121 and say that the inverse must be continuous. Thus, $f|_{U_k}$ is a homeomorphism of U_k onto U for each $k \in \mathbb{Z}$. Therefore, we do indeed have a covering map. $\square$

(b) Show that the exponential function $\exp : \mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a covering map.

We can leverage part *a*. To do this, we can use two easy facts. First, any homeomorphism produces a covering map in both directions trivially, as the pre-image can be written as the full domain, which automatically satisfies all requirements. Secondly, we can say that the product of any two covering maps is a covering map as the properties of the homeomorphism and open and closed sets and can be considered component wise. Furthemore, we can say that covering maps are preserved by the application of a homeomorphism.

From what we discussed in class, every point in $\mathbb{C}$ can be written in polar coordinates uniquely with an angle $\alpha \in [0, 2\pi)$ and a positive radius $r > 0$ except for 0, which allows us to construct this homeomorphism. From this fact, we have that $\mathbb{C}^* \cong (0, \infty) \times S^1$. We also know that we can think of $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$. Then, $e^{x+iy} = e^x e^{iy}$. Then, e^z is immediately a covering map from $\mathbb{R} \times \mathbb{R}$ to $(0, \infty) \times S^1$. This is true by our first fact that $\mathbb{R} \cong (0, \infty)$ and then for the second component, by part (a). Since, the product of any two covering maps is a covering map, and by our two homeomorphisms with $\mathbb{C}$ and $\mathbb{C}^*$, we are done. To be more explicit, this satisfies the following commutative diagram.



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