Math 246A Homework 8

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Problem 4.3

Let $\alpha, \beta : [0,1] \to \mathbb{C}$ be piecewise smooth paths, and $G : \alpha^* \times \beta^* \to \mathbb{C}$ be continuous. Then we have

$$\int_\alpha \left(\int_\beta G(z,w)\,dw \right) dz \;=\; \int_\beta \left(\int_\alpha G(z,w)\,dz \right) dw.$$

Hint: Use Problem 4.2 to show that these integrals exist and are equal.

We can consider

$$\int_\beta G(z,w)\,dw$$

By definition, this is exactly

$$\int_0^1 G(z,\beta(t))\beta'(t)\,dt$$

Now, applying the second integral,

$$\int_\alpha \left(\int_\beta G(z,w)\,dw \right) dz = \int_0^1 \left(\int_0^1 G(\alpha(s),\beta(t))\beta'(t)\,dt \right) \alpha'(s)ds = \int_0^1 \int_0^1 G(\alpha(s),\beta(t))\beta'(t)\alpha'(s)\,dt\,ds$$

For the function $F(s,t) = G(\alpha(s),\beta(t))\beta'(t)\alpha'(s)$, which is continuous, and the above integral is over the compact unit square. Since each of β, α is piecewise smooth, there exists partitions $P_t = \{0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ and $P_s = \{0 = s_0 < s_1 < \cdots < s_n = 1$ of $[0,1]$ such that the path is smooth along any $[t_k, t_{k+k}] \times [s_l \times s_{l+1}]$. We can then apply 4.2 by the smoothness, giving us

$$\sum_{\ell,k} \int_{s_\ell}^{s_{\ell+1}} \int_{t_k}^{t_{k+1}} F(s,t)\,dt\,ds = \sum_{\ell,k} \int_{t_k}^{t_{k+1}} \int_{s_\ell}^{s_{\ell+1}} F(s,t)\,ds\,dt.$$

Summing over all rectangles gives

$$\int_0^1 \int_0^1 F(s,t)\,dt\,ds = \int_0^1 \int_0^1 F(s,t)\,ds\,dt.$$

So,

$$\begin{aligned} \int_\alpha \left(\int_\beta G(z,w)\,dw \right) dz &= \int_0^1 \int_0^1 G(\alpha(s),\beta(t))\beta'(t)\alpha'(s)\,ds\,dt \\ &= \int_0^1 \left(\int_0^1 G(\alpha(s),\beta(t))\alpha'(s)\,ds \right) \beta'(t)\,dt \end{aligned}$$

$$= \int_\beta \left(\int_\alpha G(z,w)\, dz \right) dw$$

□

Problem 5.10

The periodic oscillations of a thin membrane (such as a drum) can be modeled as follows: in rest position the membrane occupies a region U in the x-y plane. Its movement under periodic (undampened) oscillations is described by a function $u = u(x, y, t)$ measuring the (small) deflection of the membrane in the z-direction in a three-dimensional x-y-z coordinate system over the point $(x, y) \in U$ at time t. The motion of the membrane is then governed by the two-dimensional wave equation

$$u_{tt} = u_{xx} + u_{yy} \tag{5.4}$$

with the boundary condition $u = 0$ on ∂U. Here we denote by a subscript differentiation with respect to the variables t, x, y. **(a)** Suppose u is a solution of (5.4) of the form $u(x, y, t) = f(t)v(x, y)$ with sufficiently smooth functions f and v, where $f \neq 0$ is a periodic function with $f(0) = 0$. Show that there exist constants $A \in \mathbb{R}$ and $\lambda > 0$ such that

$$u(x, y, t) = A \sin(\lambda t) v(x, y),$$

where

$$\Delta v + \lambda^2 v = 0. \tag{5.5}$$

First, we can differentiate to check the wave equation, We see that

$$u_{tt} = f'' v \qquad u_{xx} = f v_{xx} \qquad u_{yy} = f v_{yy}$$

Hence, by the wave equation, we can say the following when $f \neq 0$ and $v \neq 0$.

$$f'' v = f\, \Delta v \quad \implies \quad \frac{f''}{f} = \frac{\Delta v}{v} =: C$$

This value is constant since it neither depends on x, y, nor t. Then, we have a differential equation

$$f'' = Cf \qquad \Delta v = Cv$$

We know from basic facts about differential equations, we know that to obtain a periodic function f, we need $C < 0$ and then f will be a linear combination of sin and cosine functions. But, since $f(0) = 0$, we can say

$$f(t) = A \sin(\lambda t)$$

for some $\lambda, A \in \mathbb{R}$. If we differentiate twice, we see that $f''(t) = -A\lambda^2 \sin(\lambda t)$. From this and the above condition, we see that $C = -\lambda^2$.

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(b) Suppose that the membrane is circular, say $U = D$. Suppose that $u(x, y, t) = f(t)g(r)$ is a solution of (5.4), where f is as in (a) and g only depends on the radius r in polar coordinates in the x-y plane. Show that g then satisfies the differential equation

$$rg''(r) + g'(r) + \lambda^2 rg(r) = 0. \tag{5.6}$$

By question 3.9, we showed that $\Delta v = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta}$ when we express x, y in polar coordinates. Since g on depends on r, we have that $v_{\theta\theta} = 0$ so $\Delta v = v_{rr} + \frac{1}{r}v_r$. Then, as above, we can say

$$\frac{g'' + \frac{1}{r}g'}{g} = C = -\lambda^2$$

Simplifying and multiplying both sides by r produces

$$rg''(r) + g'(r) + \lambda^2 rg(r) = 0$$

□

(c) Find solutions g of (5.6) in terms of λ and the Bessel function J_0. Which condition on λ guarantees that $u(x, y, t) = f(t)g(r)$ as in (b) for one of these solutions g satisfies the boundary condition $u = 0$ on ∂D?

We know the bessel function satisfies the differential equation

$$zJ_0''(z) + J_0'(z) + zJ_0(z) = 0$$

So we need to adapt our current differential equation to be in this form. Let $x = \lambda r$ and $y(x) = g(r) = g(\frac{x}{\lambda})$. Then,

$$g'(r) = \lambda y'(x) \qquad g''(r) = \lambda^2 y''(x)$$

Then, if we plug in to the equation from (b),

$$\lambda r y''(x) + y'(x) + \lambda r y(x) = 0 \quad \Longrightarrow \quad xy''(x) + y'(x) + xy(x) = 0$$

So, $y(x) = J_0(x)$ must be a solution of the differential equation. We can also clearly have constant multiples of $J_0(x)$ as solutions. Hence, $g(r) = CJ_0(\lambda r)$ for $C \in \mathbb{C}$ All together, we can absorb the C to say,

$$u(x, y, t) = A\sin(\lambda t)J_0(\lambda r)$$

Then, since $u = 0$ on ∂D, we know that for $r = 1$, $u = 0$. So, since $A\sin(\lambda t)$ is not zero for the entire boundary, we need λ such that $J_0(\lambda) = 0$.

Problem 7.7

Let $U, V \subseteq \mathbb{C}$ be open, and $f : U \times V \to \mathbb{C}$ be a continuous function. Assume that for each $w \in V$ the function $z \mapsto f(z, w)$ is holomorphic on U, and denote by $\frac{\partial f}{\partial z}$ the derivative of this function. **(a)** Show that $\frac{\partial f}{\partial z} : U \times V \to \mathbb{C}$ is continuous.

Let's call $G(z,w) = \frac{\partial f}{\partial z}(z,w)$. We are given that for fixed w, G is holomorphic. We also know that f is continuous. Fix a ball $B(z_0,r) \subset U$ such $\overline{B(z_0,r)} \subset U$. Furthermore, fix $K \subset V$ compact. Then, we can say for $(z,w) \in \overline{B(z_0,r)} \times K$,

$$f(z,w) = \sum_{n=0}^{\infty} a_n(w)(z-z_0)^n$$

Thus, for $(z,w) \in \overline{B(z_0,r)} \times K$ again,

$$G(z,w) = \sum_{n=0}^{\infty} n a_n(w)(z-z_0)^{n-1}$$

By the theorem on power series representation of holomorphic functions, this will also converge in $B(z_0,r)$. We also know that each $a_n(w)$ is continuous, as it can be expressed below by Cauchy as an integral of a continuous function over the compact set γ^*.

$$a_n(w) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z,w)}{(z-z_0)^{n+1}}\, dz$$

It suffices now to show that G is the result of a uniformly convergent series of continuous functions. Since, f is continuous over the compact set $\gamma^* \times K$, there exists an $M \in \mathbb{N}$ such that $|f(z,w)| \leq M$ for all $(z,w) \in \gamma^* \times K$. So, by the above formula, for all $w \in K$

$$|a_n(w)| \leq \frac{M}{r^n}$$

Then, we can say for $|z-z_0| \leq \tilde{r} < r$, where G_n denotes the n-th term of the power series expansion of G,

$$|G_n(z,w)| \leq n\frac{M}{r^n}\tilde{r}^{n-1} = M\frac{n}{r}\left(\frac{\tilde{r}}{r}\right)^{n-1} =: M_n$$

We can check, by the Weierstrass M-Test, this series converges uniformly on $\overline{B(z_0,\tilde{r})} \times K$ as

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} M\frac{n}{r}\left(\frac{\tilde{r}}{r}\right)^{n-1} = \frac{M}{r}\sum_{n=1}^{\infty} n\left(\frac{\tilde{r}}{r}\right)^{n-1}$$

which converges as the derivative of a geometric series.

Hence, as each $G_n(z,w)$ is continuous since it is the product of continuous $a_n(w)$ and a polynomial, and thus the uniformly convergent series will also be continuous. Continuity is local and our choices of $B(z_0,r)$ and K were arbitrary. Hence, $G(z,w)$ is continuous on $U \times V$.

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(b) Show that if γ is a piecewise smooth path in V, then

$$z \mapsto \int_\gamma f(z,w)\, dw$$

is a holomorphic function on U and its derivative is given by

$$\frac{d}{dz}\left(\int_\gamma f(z,w)\, dw\right) = \int_\gamma \frac{\partial f}{\partial z}(z,w)\, dw.$$

So under the given assumptions we can "differentiate under the integral sign."

We first want to show $F(z) = \int_\gamma f(z,w)\,dw$ is holomorphic on U. F is clearly continuous by the standard argument that it is the integral of a continuous function over a compact domain. Now, let Δ be a closed oriented triangle in U. We can then apply question (4.3) as both $\partial\Delta$ and γ are piecewise smooth and $f(z,w)$ is continuous. We combine this with both the definition of F and the fact that the function $z \mapsto f(z,w)$ is holomorphic to use Morera's theorem.

$$\begin{aligned}
\int_{\partial\Delta} F(z)\,dz &= \int_{\partial\Delta} \left(\int_\gamma f(z,w)\,dw \right) dz \\
&= \int_\gamma \left(\int_{\partial\Delta} f(z,w)\,dz \right) dw \\
&= \int_\gamma 0\,dw \\
&= 0
\end{aligned}$$

Now, by Morera's theorem, we have that F is holomorphic. To compute the derivative, let's fix $z_0 \in U$ and $\overline{B(z_0,r)} \subset U$. We can also define the loop $\alpha = z_0 + re^{it}$ for $t \in [0, 2\pi]$. By Cauchy, we then have that

$$\begin{aligned}
F'(z_0) &= \frac{1}{2\pi i} \int_\alpha \frac{F(z)}{(z-z_0)^2}\,dz && \text{by Cauchy} \\
&= \frac{1}{2\pi i} \int_\alpha \left(\int_\gamma \frac{f(z,w)}{(z-z_0)^2}\,dw \right) dz && \text{def of } F \\
&= \int_\gamma \left(\frac{1}{2\pi i} \int_\alpha \frac{f(z,w)}{(z-z_0)^2}\,dz \right) dw && \text{by (4.3)} \\
&= \int_\gamma \frac{\partial f}{\partial z}(z_0,w)\,dw && f \text{ holomorphic in } z
\end{aligned}$$

Thus, differentiating under the integral sign exactly gives us the derivative we want.

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(c) Assume in addition that the function f is also holomorphic in its second variable, i.e., for fixed $z \in U$ the map $w \mapsto f(z,w)$ is holomorphic on V. Show that for each $z \in U$ the map $w \mapsto \frac{\partial f}{\partial z}(z,w)$ is holomorphic on V.

Let's take the same set up as in part (b) but with Δ a closed oriented triangle in V. By part (b), we can say

$$\int_{\partial\Delta} \frac{\partial f}{\partial z}(z,w)\,dw = \frac{d}{dz}\left(\int_{\partial\Delta} f(z,w)\,dw \right)$$

But, as we just assumed the map $w \mapsto f(z,w)$ is holomorphic on V, for each closed oriented triangle, we have the following by Morera's Theorem

$$\int_{\partial\Delta} f(z,w)\,dw = 0$$

Hence,

$$\int_{\partial\Delta} \frac{\partial f}{\partial z}(z,w)\,dw = 0$$

Thus, we are done by Morera's as each $\frac{\partial f}{\partial z}(z, w)$ is continuous $U \times V$. $\square$

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Problem 8.1

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and suppose that $f(\sqrt[n]{n}) \in \mathbb{R}$ for all $n \in \mathbb{N}$. Show that $f(\mathbb{R}) \subseteq \mathbb{R}$.

Since f is entire, it can represented everywhere as a power series, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let's define $g(z) = \overline{f(\overline{z})}$. This produces $g(z) = \sum_{n=0}^{\infty} \bar{a_n} z^n$. Hence, g is entire as we can represent it as a power series that must converge everywhere as the power series for f does. . Furthermore, by the fact that we are given $f(\sqrt[n]{n}) \in \mathbb{R}$, we have

$$f(\sqrt[n]{n}) = \overline{f(\overline{\sqrt[n]{n}})} = g(\sqrt[n]{n})$$

By the uniqueness theorem, it suffices to show that the set $S = \{\sqrt[n]{n} \mid n \in \mathbb{N}\}$ has a limit point. But this is immediate as the limit as $n \to \infty$ of $\sqrt[n]{n}$ is 1 and $1 \in S$ since the first root of 1 is 1. Thus, by the uniqueness theorem, $f(z) = g(z)$ for all $z \in \mathbb{C}$. Hence, $a_n = \overline{a_n}$ for all $n \in \mathbb{N}$. Thus, we can see that for real z, $f(z) \in \mathbb{R}$. Thus, $f(\mathbb{R}) \subseteq \mathbb{R}$.

$\square$

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