Math 131A Notes

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Definition (Algebraic Number)**.** A complex number $\alpha \in \mathbb{C}$ is called an *algebraic number* if there exists a non-zero polynomial with integer coefficients

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{Z}, \quad a_n \neq 0,$$

such that $p(\alpha) = 0$. In other words, α is a root of a polynomial with integer coefficients.

The set of all algebraic numbers is denoted by $\overline{\mathbb{Q}}$ or A.

Theorem (Rational Zeros Theorem)**.** Suppose $c_0, c_1, \ldots, c_n$ are integers and r is a rational number satisfying the polynomial equation

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $n \geq 1$, $c_n \neq 0$, and $c_0 \neq 0$. Let $r = \frac{c}{d}$ where c, d are integers having no common factors and $d \neq 0$. Then c divides c_0 and d divides c_n.

In other words, the only rational candidates for solutions of the polynomial equation have the form $\frac{c}{d}$, where c divides c_0 and d divides c_n.

Definition (Radicals are not in $\mathbb{Q}$)**.** *Example 3:* $\sqrt{17}$ is not a rational number.

Proof: The only possible rational solutions of the equation

$$x^2 - 17 = 0$$

are $\pm 1, \pm 17$. None of these numbers are solutions, and thus $\sqrt{17}$ is not a rational number.

Definition (Order on $\mathbb{Q}$)**.** The set $\mathbb{Q}$ also has an order structure $\leq$ satisfying the following properties:

- O1. Given a and b, either $a \leq b$ or $b \leq a$.
- O2. If $a \leq b$ and $b \leq a$, then $a = b$.
- O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.
- O4. If $a \leq b$, then $a + c \leq b + c$.
- O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Definition (Consequences of the Field Properties)**.** The following are consequences of the field properties for $a, b, c \in \mathbb{R}$:

- (i) $a + c = b + c$ implies $a = b$.
- (ii) $a \cdot 0 = 0$ for all a.

(iii) $(-a)b = -ab$ for all a, b.

(iv) $(-a)(-b) = ab$ for all a, b.

(v) $ac = bc$ and $c \neq 0$ imply $a = b$.

(vi) $ab = 0$ implies either $a = 0$ or $b = 0$.

Definition (Consequences of the Properties of an Ordered Field)**.** The following are consequences of the properties of an ordered field for $a, b, c \in \mathbb{R}$:

(i) If $a \leq b$, then $-b \leq -a$.

(ii) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.

(iii) If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

(iv) $0 \leq a^2$ for all a.

(v) $0 < 1$.

(vi) If $0 < a$, then $0 < a^{-1}$.

(vii) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Note: $a < b$ means $a \leq b$ and $a \neq b$.

Theorem (Triangle Inequality and Misc)**.** The following properties hold for the absolute value function for $a, b \in \mathbb{R}$:

(i) $|a| \geq 0$ for all $a \in \mathbb{R}$.

(ii) $|ab| = |a| \cdot |b|$ for all $a, b \in \mathbb{R}$.

(iii) $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$ (Triangle Inequality).

Corollary (Consequence of the Triangle Inequality)**.** The following property holds for the absolute value function for $a, b \in \mathbb{R}$:

$$||a| - |b|| \leq |a - b|$$

1 Completeness

Definition (Bounded Definitions)**.** Let $\emptyset \neq A \subseteq \mathbb{R}$.

1. We say that A is *bounded above* if there exists $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in A$. In this case, M is called an *upper bound* for A. If moreover $M \in A$, then M is called the *maximum* of A.
2. We say that A is *bounded below* if there exists $m \in \mathbb{R}$ such that $m \leq a$ for all $a \in A$. In this case, m is called a *lower bound* for A. If moreover $m \in A$, then m is called the *minimum* of A.
3. We say that A is *bounded* if it is both bounded below and bounded above.

Definition (Supremum and Infimum)**.** Let $\emptyset eq A \subseteq \mathbb{R}$.

1. Let A be bounded above. We say L is a *least upper bound* for A if:
   (a) L is an upper bound for A.
   (b) If M is an upper bound for A, then $L \leq M$.

   This L is also called the *supremum* of A and we write $L = \sup A$.

2. Let A be bounded below. We say ℓ is a *greatest lower bound* for A if:
   (a) ℓ is a lower bound for A.
   (b) If m is a lower bound for A, then $m \leq \ell$.

   This ℓ is also called the *infimum* of A and we write $\ell = \inf A$.

Definition (Least Upper Bound and Greatest Lower Bound Properties)**.** Let $\emptyset \neq S \subseteq \mathbb{R}$.

1. We say S has the *least upper bound property* if for every nonempty subset A of S which is also bounded above, A has a least upper bound in S.
2. We say S has the *greatest lower bound property* if for every nonempty subset A of S which is also bounded below, A has a greatest lower bound in S.

Theorem (Axiom of $\mathbb{R}$)**.** The set of real numbers $\mathbb{R}$ has the least upper bound property. In fact, it is the unique ordered field with the least upper bound property. As a corollary, the set of real numbers $\mathbb{R}$ has the greatest lower bound property.

Property (Archimedean Property of $\mathbb{R}$)**.** For any $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $x < n$. This n depends on x.

Proof. Proof by contradiction. Suppose not, then there exists $x \in \mathbb{R}$ such that $x \geq n$ for all $n \in \mathbb{N}$. Hence, $\mathbb{N} \subseteq \mathbb{R}$ is bounded above. By the least upper bound property of $\mathbb{R}$, we have $\sup \mathbb{N} = L$ exists in $\mathbb{R}$. Then $L - 1$ is not an upper bound for $\mathbb{N}$, so there is an $m \in \mathbb{N}$ such that $m > L - 1$. But then $m + 1 \in \mathbb{N}$ and $m + 1 > L$, contradicting $L = \sup \mathbb{N}$.

□

Corollary (AP Corollary)**.** If $a > 0$, $b > 0$, then there exists $n \in \mathbb{N}$ such that $na > b$.

Corollary (AP Corollary)**.** For $a \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $n \leq a < n + 1$.

Proof. If $a \in \mathbb{Z}$, take $n = a$.

For $a > 0$ and $a \notin \mathbb{N}$, define $S = \{n \in \mathbb{Z} : n < a\} \ni 0$. We claim that there is an $m \in \mathbb{Z}$ such that $m \in S$ but $m + 1 \notin S$. If not, $m \in S$ implies $m + 1 \in S$, and we have $0 \in S$, thus by induction $\mathbb{N} \cup \{0\} \subseteq S$. This implies $\mathbb{N}$ is bounded above as S is, which is a contradiction. Take $n = m$.

For non-integer $a < 0$, we have $-a > 0$. Then there is $\ell \in \mathbb{N}$ such that $\ell < -a < \ell + 1$, and so $-\ell - 1 < a \leq -\ell$. Take $n = -\ell - 1$.

□

Corollary (AP flipped)**.** For $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \epsilon$.

Definition (Density in $\mathbb{R}$)**.** Let set $A \subseteq \mathbb{R}$ be called *dense in* $\mathbb{R}$ if for any $x, y \in \mathbb{R}$ with $x < y$, there exists an $a \in A$ such that $x < a < y$.

Theorem (Rationals Dense in Reals)**.** The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. Then there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$. There exists $m \in \mathbb{Z}$ such that $m - 1 \leq nx < m$. Then

$$\frac{m-1}{n} \leq x < \frac{m}{n}$$

and so

$$x < \frac{m}{n} \leq x + \frac{1}{n} < y,$$

noting that $\frac{m}{n} \in \mathbb{Q}$. □

Corollary (Irrationals Dense in Reals)**.** The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. Then $x\sqrt{2} < y\sqrt{2}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $x\sqrt{2} < r < y\sqrt{2}$, which implies $x < \sqrt{2}r < y$. Note that $\sqrt{2}r \in \mathbb{R} \setminus \mathbb{Q}$. □

Definition (Extension to Infinity)**.** The symbols $+\infty$, $-\infty$. We adjoin these symbols with $\mathbb{R}$ so that $-\infty < a < +\infty$ for all $a \in \mathbb{R}$. If $\emptyset eq A \subseteq \mathbb{R}$ is not bounded above, we set $\sup A = +\infty$. Similarly, if $\emptyset eq A \subseteq \mathbb{R}$ is not bounded below, we set $\inf A = -\infty$.

Definition (Sequences of Real Numbers)**.** A *sequence of real numbers* is a function $f : \mathbb{N} o \mathbb{R}$. We can represent this function f as

$$f(1), f(2), \ldots$$

or $(f(n))_{n \in \mathbb{N}}$, or more commonly $(f_n)_{n \in \mathbb{N}}$, $(f_n)_{n \geq 1}$, or simply (f_n). We can also use curly braces, such as $\{f_n\}$, to denote the sequence.

Examples:

1. $(a_n)_{n \in \mathbb{N}}$ with $a_n = \frac{1}{n}$
2. $(a_n)_{n \in \mathbb{N}}$ with $a_n = (-1)^n$
3. $(a_n)_{n \in \mathbb{N}}$ with $a_n = n^2$
4. $(a_n)_{n \in \mathbb{N}}$ with $a_n = \cos\left(\frac{n\pi}{2}\right)$

2 Limits and Convergence

Definition (Convergence of a Sequence)**.** A sequence (a_n) of real numbers *converges* if there exists $a \in \mathbb{R}$ such that for any given $\epsilon > 0$, there exists an $n_\epsilon \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq n_\epsilon$.

In this case, a is called the *limit* of the sequence, and we write

$$a = \lim_{n\to\infty} a_n$$

or $a_n \to a$ as $n \to \infty$. We say (a_n) *converges to* a. If no such limit a exists, i.e., if the sequence does not converge, then we say the sequence *diverges.*

Theorem (Uniqueness of Limit)**.** The limit of a sequence is unique.

Proof. Assume (a_n) converges and $\lim_{no\infty} a_n = a$ and $\lim_{no\infty} a_n = b$. We want to show $a = b$.

Let $\epsilon > 0$. There exist $n_1, n_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2}$ for all $n \geq n_1$ and $|a_n - b| < \frac{\epsilon}{2}$ for all $n \geq n_2$. Then for $n \geq \max(n_1, n_2)$, we have $|a_n - a| < \frac{\epsilon}{2}$ and $|a_n - b| < \frac{\epsilon}{2}$.

Therefore, with such n, we have

$$|a - b| \leq |a - a_n| + |a_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude $a = b$. □

Example (Limit Examples)**. Example 1** Show that (a_n) with $a_n = \frac{1}{n}$ converges to zero.

Proof. Let $\epsilon > 0$, we need to find $n_\epsilon \in \mathbb{N}$ such that $|a_n - 0| = a_n < \epsilon$ for all $n \geq n_\epsilon$. By the Archimedean property of $\mathbb{R}$, there exists $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon > \frac{1}{\epsilon}$. Then for $n \geq n_\epsilon$, we have

$$\frac{1}{n} \leq \frac{1}{n_\epsilon} < \epsilon.$$

□

Example 2 Show that (a_n) with $a_n = (-1)^n$ diverges.

Proof. By contradiction. Suppose $a_n \to a \in \mathbb{R}$. Then $|a_n - a| < \frac{1}{2}$ for all $n \geq m$ for some $m \in \mathbb{N}$. For even $n \geq m$, we have $|1 - a| < \frac{1}{2}$, and for odd $n \geq m$, we have $|-1 - a| < \frac{1}{2}$. Then

$$2 = 1 + a + 1 - a \leq |1 + a| + |1 - a| < 1,$$

which is a contradiction. □

Example 3 Show that $\lim_{n\to\infty} \frac{3n+1}{5n-2} = \frac{3}{5}$.

Proof. Let $\epsilon > 0$. It is enough to show there exists $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$, we have

$$\left| \frac{3n+1}{5n-2} - \frac{3}{5} \right| < \epsilon,$$

i.e.,

$$\frac{11}{5(5n-2)} < \epsilon.$$

Note that

$$\frac{11}{5\epsilon} < 5n - 2 \iff n > \frac{2}{5} + \frac{11}{25\epsilon}.$$

So choose $n_\epsilon \in \mathbb{N}$ satisfying

$$n_\epsilon > \frac{2}{5} + \frac{11}{25\epsilon}.$$

Then for all $n \geq n_\epsilon$, we have

$$n > \frac{2}{5} + \frac{11}{25\epsilon},$$

which implies

$$\epsilon > \frac{11}{5(5n-2)} = \frac{11}{5(5n-2)}.$$

□

Theorem (Convergent Sequences are Bounded). Convergent sequences are bounded.

Proof. Let (a_n) be a convergent sequence converging to $a \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that $|a_n - a| < 1$ for all $n \geq N$. Thus $|a_n| \leq |a_n - a| + |a| < 1 + |a|$ for all $n \geq N$.

Let $M = \max\{|a_1|, \ldots, |a_{N-1}|, 1 + |a|\}$, then for all $n \in \mathbb{N}$, $|a_n| \leq M$. □ □

Theorem (Limit Properties). Let $(a_n), (b_n)$ be two convergent sequences with limits $a, b \in \mathbb{R}$. Then:

1. For $k \in \mathbb{R}$, we have $\lim_{n\to\infty} ka_n = ka$.
2. $\lim_{n\to\infty}(a_n + b_n) = a + b$.
3. $\lim_{n\to\infty} a_n b_n = ab$.
4. If $a_n \neq 0$ for all $n \in \mathbb{N}$ and $a \neq 0$, then $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.
5. If $a_n \neq 0$ for all $n \in \mathbb{N}$ and $a \neq 0$, then $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{a}$.

Proof.

1. Let $\epsilon > 0$. Then there exists $n_\epsilon \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{k}$, which implies $|ka_n - ka| < \epsilon$.
2. Let $\epsilon > 0$. Then there exist $n_1, n_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2}$ and $|b_n - b| < \frac{\epsilon}{2}$. Then for $n \geq n_\epsilon = \max(n_1, n_2)$, we have $|(a_n + b_n) - (a + b)| < \epsilon$.
3. Let $\epsilon > 0$. We want to find $n_\epsilon \in \mathbb{N}$ such that $|a_n b_n - ab| < \epsilon$. Let M be such that $|a_n| \leq M$ for all n. Let n_1, n_2 be such that $|a_n - a| < \frac{\epsilon}{2(|b|+1)}$ for all $n \geq n_1$ and $|b_n - b| < \frac{\epsilon}{2M}$. Then
$$|a_n b_n - ab| \leq |a_n||b_n - b| + |b||a_n - a| < M \cdot \frac{\epsilon}{2M} + |b| \cdot \frac{\epsilon}{2(|b|+1)} < \epsilon.$$
4. Claim: $\inf\{|a_n| : n \in \mathbb{N}\} = m > 0$. Indeed, there is n_1 such that for all $n \geq n_1$, one has $|a_n - a| < \frac{|a|}{2}$, which implies $|a_n| \geq |a| - |a_n - a| \geq \frac{|a|}{2}$. So $m = \inf_n |a_n| \geq \inf\{|a_1|, \ldots, |a_{n_1}|, \frac{|a|}{2}\} > 0$.
   Now choose $n_\epsilon \in \mathbb{N}$ such that $|a_n - a| < \epsilon |a| m$. Then
$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \frac{|a_n - a|}{|a_n||a|} < \epsilon.$$
5. Combine (3) and (4).

□

Definition (Extension of Limits to Infinity). For a sequence (s_n), we write $\lim s_n = +\infty$ provided that for each $M > 0$, there is a number N such that $n > N$ implies $s_n > M$.

In this case, we say the sequence *diverges to* $+\infty$.

Similarly, we write $\lim s_n = -\infty$ provided that for each $M < 0$, there is a number N such that $n > N$ implies $s_n < M$.

Example (Divergence to Infinity)**.** We need to consider an arbitrary $M > 0$ and show there exists N (which will depend on M) such that $n > N$ implies $\sqrt{n} + 7 > M$.

To see how big N must be, we "solve" for n in the inequality $\sqrt{n}+7 > M$. This inequality holds provided $\sqrt{n} > M - 7$ or $n > (M-7)^2$. Thus, we will take $N = (M-7)^2$.

Formal Proof.

Let $M > 0$ and let $N = (M-7)^2$. Then $n > N$ implies $n > (M-7)^2$, hence $\sqrt{n} > M - 7$, hence $\sqrt{n} + 7 > M$. This shows $\lim(\sqrt{n} + 7) = +\infty$.

Theorem. Let (s_n) and (t_n) be sequences such that

$$\lim_{n\to\infty} s_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} t_n > 0$$

(where $\lim_{n\to\infty} t_n$ can be finite or $+\infty$). Then

$$\lim_{n\to\infty} s_n t_n = +\infty.$$

Proof. Let $M > 0$ be given. Choose a real number m such that

$$0 < m < \lim_{n\to\infty} t_n.$$

Such an m exists because $\lim_{n\to\infty} t_n > 0$.

There are two cases to consider:

1. **Case 1** $\lim_{n\to\infty} t_n$ is finite.

   Since $\lim_{n\to\infty} t_n > m$, there exists an integer N_1 such that for all $n > N_1$,

   $$t_n > m.$$

2. **Case 2:** $\lim_{n\to\infty} t_n = +\infty$.

   In this scenario, $t_n > m$ holds for all sufficiently large n, so we can similarly find an integer N_1 such that for all $n > N_1$,

   $$t_n > m.$$

Since $\lim_{n\to\infty} s_n = +\infty$, there exists an integer N_2 such that for all $n > N_2$,

$$s_n > \frac{M}{m}.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n > N$,

$$s_n t_n > \frac{M}{m} \cdot m = M.$$

Since M was arbitrary, it follows that $\lim_{n\to\infty} s_n t_n = +\infty$. □

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3 Sequences Post Midterm

Definition (Limit Superior)**.** Let (a_n) be a sequence of real numbers. The **limit superior** (or $\limsup$) of (a_n) is defined by:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k.$$

This is the greatest limit point of the sequence, or equivalently, the largest value to which any subsequence of (a_n) converges.

Definition (Limit Inferior)**.** Let (a_n) be a sequence of real numbers. The **limit inferior** (or $\liminf$) of (a_n) is defined by:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k.$$

This is the smallest limit point of the sequence, or equivalently, the lowest value to which any subsequence of (a_n) converges.

Definition. A sequence (a_n) of real numbers is called:

1. **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$,
2. **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$,
3. **strictly increasing** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$,
4. **strictly decreasing** if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

A sequence that is either increasing or decreasing is called a **monotone sequence**.

Theorem. All bounded monotone sequences converge.

Proof. Let (s_n) be a bounded increasing sequence. Let S denote the set $\{s_n : n \in \mathbb{N}\}$, and let $u = \sup S$. Since S is bounded, u represents a real number. We show $\lim s_n = u$.

Let $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for S, there exists s_N such that $s_N > u - \epsilon$. Since (s_n) is increasing, we have $s_N \leq s_n$ for all $n \geq N$. Of course, $s_n \leq u$ for all n, so $n > N$ implies $u - \epsilon < s_n \leq u$, which implies $|s_n - u| < \epsilon$. This shows $\lim s_n = u$.

The proof for bounded decreasing sequences is left to Exercise 10.2. $\square$

Definition. Let (a_n) be a bounded sequence (convergent or not). Then the limiting behavior of (a_n) depends on the set of the form

$$\{a_n : n \geq N\} = \bigcup_N A_N.$$

Let us define

$$u_N = \inf\{a_n : n \geq N\} \quad \text{and} \quad v_N = \sup\{a_n : n \geq N\}.$$

Then

$$u_1 \leq u_2 \leq \cdots \leq u_N \leq \cdots \quad \text{and} \quad v_1 \geq v_2 \geq \cdots \geq v_N \geq \cdots,$$

i.e., (u_N) is increasing and (v_N) is decreasing.

Definition (Cauchy Sequences)**.** A sequence (a_n) in $\mathbb{R}$ is called **Cauchy** if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon \quad \text{for all } n, m \geq N.$$

Proposition. Convergent sequences are Cauchy.

Proof. Let (a_n) be a Cauchy sequence in $\mathbb{R}$ converging to $a \in \mathbb{R}$. Let $\epsilon > 0$. Then $\epsilon/2 > 0$ and hence there exists $N_{\epsilon/2} \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2} \quad \text{for all } n \geq N_{\epsilon/2}.$$

Let $n, m \geq N_{\epsilon/2}$. Then

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, (a_n) is a Cauchy sequence. □

Lemma. Cauchy sequences are bounded.

Proof. Let (a_n) be a Cauchy sequence. Then there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$.

Let

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}.$$

Then for all $n = 1, \ldots, N-1$, we have $|a_n| \leq M$. For all $n \geq N$, we have

$$|a_n| \leq |a_N| + 1 \leq M.$$

Hence, for all $n \in \mathbb{N}$, we have $|a_n| \leq M$. □

Theorem. A sequence (a_n) in $\mathbb{R}$ converges if and only if (a_n) in $\mathbb{R}$ is Cauchy.

Proof. $(\Rightarrow)$ This direction follows from the above proposition.

$(\Leftarrow)$ Since (a_n) is Cauchy, it is bounded. Then it is enough to check that $\limsup a_n = \liminf a_n$.

Let $\epsilon > 0$. Then there exists $n_2 \in \mathbb{N}$ such that for all $n, m \geq n_2$,

$$|a_n - a_m| < \epsilon.$$

This implies:

$$a_n < a_m + \epsilon \quad \text{for all } n, m \geq n_2,$$
$$u_{n_2} \leq a_m + \epsilon \quad \text{for all } m \geq n_2,$$
$$u_{n_2} \leq u_{n_2} + \epsilon,$$
$$\limsup a_n \leq u_{n_2} \leq u_{n_2} + \epsilon \leq \liminf a_n + \epsilon,$$

i.e.,

$$\limsup a_n \leq \liminf a_n.$$

Since $\epsilon > 0$ is arbitrary, we have $\limsup a_n = \liminf a_n$. □

Definition. Let (k_n) be a sequence of natural numbers such that $k_{n+1} > k_n$ for all $n \in \mathbb{N}$. Let (a_n) be a sequence of real numbers. Then the sequence $(a_{k_n})_{n\in\mathbb{N}}$ is called a **subsequence** of $(a_n)_{n\in\mathbb{N}}$.

Remark. It is easy to see that $k_n \geq n$ for all $n \in \mathbb{N}$ and hence $k_n \to \infty$ as $n \to \infty$.

Theorem. Every sequence has a monotone subsequence.

Proof. We say that the nth term is the **dominant** term if $a_n > a_m$ for $m > n$.

Case 1: There are infinitely many dominant terms. Let a_{k_1} be the first dominant term, a_{k_2} the next, and so on. Then clearly, (a_{k_n}) is a decreasing subsequence of (a_n).

Case 2: There are only finitely many dominant terms. Let a_M be the last dominant term in the sequence $a_1, a_2, \ldots$. Let $k_1 > M$. Since a_{k_1} is not dominant, there exists $k_2 > k_1$ such that $a_{k_2} \geq a_{k_1}$. Since a_{k_2} is not dominant, there exists $k_3 > k_2$ such that $a_{k_3} \geq a_{k_2}$.

We proceed inductively to construct a subsequence (a_{k_n}) of (a_n): Assume we have found $k_1, \ldots, k_m$ such that

$$a_{k_1} \leq a_{k_2} \leq \cdots \leq a_{k_m}.$$

Then, since a_{k_m} is not dominant, there exists $k_{m+1} > k_m$ such that

$$a_{k_{m+1}} \geq a_{k_m}.$$

Thus, we have constructed an increasing subsequence. □

Theorem (Bolzano-Weierstrass)**.** Any bounded sequence has a convergent subsequence.

Proof. Let (a_n) be a bounded sequence. Then by the previous theorem, it has a monotone subsequence, say (a_{n_k}). Since (a_{n_k}) is bounded and monotone, it is convergent. □

Theorem. Let (a_n) be a sequence in $\mathbb{R}$. Then:

(a) There exists a subsequence whose limit is $\limsup_{n\to\infty} a_n$.

(b) There exists a subsequence whose limit is $\liminf_{n\to\infty} a_n$.

Definition. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. A **subsequential limit** is any $a \in \mathbb{R} \cup \{-\infty, \infty\}$ that is the limit of any subsequence of $\{a_n\}$.

Remark. If $\lim_{n\to\infty} a_n = a$, then the set of all subsequential limits is $\{a\}$.

Theorem. Let $\{a_n\} \subseteq \mathbb{R}$ and let A be the set of all subsequential limits of $\{a_n\}$. Then:

(a) $A \neq \emptyset$.

(b) $\sup A = \limsup_{n\to\infty} a_n$ and $\inf A = \liminf_{n\to\infty} a_n$.

(c) $\lim_{n\to\infty} a_n$ exists if and only if $|A| = 1$.

Theorem. Let A denote the set of all subsequential limits of (a_n). Suppose $\{b_n\}$ is a sequence in $A \cap \mathbb{R}$ with $b = \lim_{n\to\infty} b_n$. Then $b \in A$.

Theorem. If $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ and $a > 0$, and (b_n) is a sequence in $\mathbb{R}$, then

$$\limsup(a_n b_n) = a \limsup b_n.$$

Theorem. Let (a_n) be a sequence of real numbers. Then

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf a_n^{1/n} \leq \limsup a_n^{1/n} \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|.$$

Definition (Alternative Definitions of Limit Superior)**.** Let (a_n) be a sequence of real numbers. The **limit superior** ($\limsup$) can be equivalently defined as:

- **Supremum of Subsequence Limits:**

$$\limsup_{n\to\infty} a_n = \sup \left\{ \ell \in \mathbb{R} \;\middle|\; \text{there exists a subsequence } (a_{n_k}) \text{ such that } \lim_{k\to\infty} a_{n_k} = \ell \right\}.$$

- **Infimum of Supremums of Tails:**

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1} \left(\sup_{k\geq n} a_k \right).$$

- **Eventually Upper Bounds:** For every $\epsilon > 0$, there exists an N such that for all $n \geq N$,

$$a_n \leq \limsup_{k\to\infty} a_k + \epsilon,$$

  and there are infinitely many n for which

$$a_n \geq \limsup_{k\to\infty} a_k - \epsilon.$$

- **Using Negation and Limit Inferior:**

$$\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty}(-a_n).$$

Definition (Alternative Definitions of Limit Inferior)**.** Let (a_n) be a sequence of real numbers. The **limit inferior** ($\liminf$) can be equivalently defined as:

- **Infimum of Subsequence Limits:**

$$\liminf_{n\to\infty} a_n = \inf \left\{ \ell \in \mathbb{R} \;\middle|\; \text{there exists a subsequence } (a_{n_k}) \text{ such that } \lim_{k\to\infty} a_{n_k} = \ell \right\}.$$

- **Supremum of Infimums of Tails:**

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1} \left(\inf_{k\geq n} a_k \right).$$

- **Eventually Lower Bounds:** For every $\epsilon > 0$, there exists an N such that for all $n \geq N$,

$$a_n \geq \liminf_{k\to\infty} a_k - \epsilon,$$

  and there are infinitely many n for which

$$a_n \leq \liminf_{k\to\infty} a_k + \epsilon.$$

- **Using Negation and Limit Superior:**

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n).$$

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4 Series

Definition. Let (a_n) be a sequence in $\mathbb{R}$. For $n \in \mathbb{N}$, define $s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$. The series $\sum_{n=1}^{\infty} a_n$ is said to **converge** if the sequence of partial sums (s_n) converges. A series that does not converge is said to **diverge**.

Definition. A series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges. [Note that the series $\sum_{n=1}^{\infty} |a_n|$ either converges or diverges to ∞.]

Theorem. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if it satisfies the Cauchy criterion, i.e., for $\forall \epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ such that

$$\left| \sum_{k=n}^{n+m} a_k \right| < \epsilon \quad \text{for all } n \geq N_\epsilon \text{ and } m \in \mathbb{N} \cup \{0\}.$$

Proof. $\sum_{n=1}^{\infty} a_n$ converges $\iff$ (s_n) converges $\iff$ (s_n) is Cauchy.
$\iff$ $\forall \epsilon > 0, \exists \tilde{N}_\epsilon \in \mathbb{N}$ such that $|s_m - s_n| < \epsilon$ for all $m, n \geq \tilde{N}_\epsilon$.
$\iff$ $\forall \epsilon > 0, \exists \tilde{N}_\epsilon \in \mathbb{N}$ such that $\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon$ for all $n \geq \tilde{N}_\epsilon, m \in \mathbb{N}$.
$\iff$ $\forall \epsilon > 0, \exists \tilde{N}_\epsilon \in \mathbb{N}$ such that $|\sum_{k=n}^{\infty} a_k| < \epsilon$ for all $n \geq \tilde{N}_\epsilon$ and $m \in \mathbb{N} \cup \{0\}$. □

Corollary. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem (Comparison test)**.** Let (a_n) be a series with $a_n \geq 0$ for all $n \in \mathbb{N}$.

1. If $\sum_{n=1}^{\infty} a_n$ converges and $|b_n| \leq a_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} b_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n = \infty$ and $b_n \geq a_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} b_n = \infty$.

Proof.

1. This follows from the fact

$$\left| \sum_{k=n}^{n+m} b_k \right| \leq \sum_{k=n}^{n+m} a_k$$

and the Cauchy criterion.

2. Since $a_n \leq b_n$ for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} b_k \geq \sum_{k=1}^{n} a_k.$$

Since $\sum_{k=1}^{n} a_k \to \infty$ as $n \to \infty$, we have $\sum_{k=1}^{n} b_k \to \infty$ as $n \to \infty$.

□

Theorem (The root test)**.** Let $\sum_{n=1}^{\infty} a_n$ be a series and let

$$\alpha = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Then the series $\sum_{n=1}^{\infty} a_n$:

1. converges absolutely if $\alpha < 1$,
2. diverges if $\alpha > 1$,

3. has no conclusion if $\alpha = 1$.

Proof. 1. Let $\epsilon > 0$ be such that $\alpha + \epsilon < 1$. Since

$$\alpha = \limsup_{n\to\infty} |a_n|^{1/n} = \inf_N \sup_{n\geq N} |a_n|^{1/n},$$

there exists $N_\epsilon \in \mathbb{N}$ such that

$$\sup\{|a_n|^{1/n} : n \geq N_\epsilon\} < \alpha + \epsilon.$$

This implies

$$|a_n| < (\alpha + \epsilon)^n \quad \text{for all } n \geq N_\epsilon.$$

Since $\sum_{n=1}^\infty (\alpha + \epsilon)^n$ converges, we conclude that $\sum_{n=1}^\infty |a_n|$ converges.

2. There exists a subsequence (a_{k_n}) of (a_n) such that $\lim_{n\to\infty} |a_{k_n}|^{1/k_n} = \alpha > 1$. This implies there exists $N \in \mathbb{N}$ such that

$$|a_n|^{1/n} > 1 \quad \text{for all } n \geq N,$$

which leads to

$$|a_n| > 1 \quad \text{for all } n \geq N.$$

This implies $\lim_{n\to\infty} a_n \neq 0$ and the Cauchy criterion is not satisfied. Thus, $\sum_{n=1}^\infty a_n$ does not converge. □

Theorem (The ratio test)**.** Let $\sum_{n=1}^\infty a_n$ be a series with $a_n \neq 0$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^\infty a_n$:

1. converges absolutely if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,
2. diverges if $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$,
3. has no conclusion if $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Proof. Recall that

$$\liminf_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Parts 1 and 2 follow from the root test. Part 3 uses the same counterexamples as in the root test. □

Theorem (Abel's criterion)**.** Let (a_n) be a decreasing sequence with $\lim_{n\to\infty} a_n = 0$. Let (b_n) be such that $\sum_{k=1}^\infty b_k$ is bounded. Then $\sum_{n=1}^\infty a_n b_n$ converges.

Proof. Let $t_n = \sum_{k=1}^n b_k$. As (t_n) is bounded, there exists $M > 0$ such that $|t_n| \leq M$ for all $n \in \mathbb{N}$.

For $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $|a_n| < \frac{\epsilon}{2M}$ for all $n \geq N_\epsilon$. Now,

$$\sum_{k=n}^{n+m} a_k b_k = \sum_{k=n}^{n+m} a_k (t_k - t_{k-1}) = \sum_{k=n}^{n+m} (a_k - a_{k+1}) t_k + a_{n+m+1} t_{n+m} - a_n t_{n-1}.$$

This implies

$$\left|\sum_{k=n}^{n+m} a_k b_k\right| \leq M(a_n - a_{n+m+1}) + M a_{n+m+1} + M a_n < \epsilon \quad \text{for all } n \geq N_\epsilon \text{ and } m \in \mathbb{N} \cup \{0\}.$$

□

Corollary (Leibniz Criterion)**.** If (a_n) is decreasing and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^\infty (-1)^n a_n$ converges.

Theorem (The dyadic criterion)**.** Let (a_n) be decreasing and $a_n \geq 0$. Then $\sum a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Proof. Let $s_n = \sum_{k=0}^{n} a_k$ and $t_n = \sum_{k=0}^{n} 2^k a_{2^k}$. Then:

$$2^n a_{2^{n+1}} \leq \sum_{k=2^n}^{2^{n+1}-1} a_k \leq 2^n a_{2^n}.$$

Using the fact that (a_n) is decreasing:

$$2^{n+1} a_{2^{n+1}} \leq 2 \sum_{k=n}^{\infty} 2^k a_{2^k}.$$

This implies:

$$\sum_{k=n}^{\infty} 2^{k+1} a_{2^k} \leq 2 \sum_{k=n}^{\infty} 2^k a_k.$$

Hence:

$$t_{n+1} - a_1 \leq s_{2^{n+1}} \quad \text{and} \quad s_n \leq t_n + a_1.$$

Since (s_n) converges if and only if (t_n) converges, we conclude:

$$\sum a_n \text{ converges if and only if } \sum_{n=0}^{\infty} 2^n a_{2^n} \text{ converges.}$$

□

Theorem (The Raab-Duhamel Criterion)**.** Let $\sum a_n$ be a series with $a_n > 0 \; \forall n \in \mathbb{N}$. Suppose that $\exists n_0 \in \mathbb{N}$ and $q > 1$ such that

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq q \quad \forall n \geq n_0.$$

Then the series $\sum_n a_n$ converges.

Proof. Let $q = 1 + \varepsilon$ for some $\varepsilon > 0$. Then

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq q = 1 + \varepsilon$$

implies

$$na_n - na_{n+1} \geq a_{n+1} + \varepsilon a_{n+1}.$$

Rearranging, we have

$$na_n \geq (n+1)a_{n+1} + \varepsilon a_{n+1} \quad \forall n \geq n_0.$$

Thus,

$$\begin{aligned} n_0 a_{n_0} &\geq (n+1)a_{n+1} + \varepsilon a_{n+1} + (n+2)a_{n+2} + \varepsilon a_{n+2} + \cdots \\ &\geq (n+p)a_{n+p} + \varepsilon \sum_{k=n+1}^{n+p} a_k \quad \forall n \in \mathbb{N}. \end{aligned}$$

This implies

$$a_{n_0+1} + \cdots + a_{n_0+m} \leq \frac{n_0 a_{n_0}}{\varepsilon} \quad \forall n \in \mathbb{N}.$$

Therefore, the partial sum of $\sum_{n=n_0+1}^{\infty} a_n$ is bounded above by $\frac{n_0 a_{n_0}}{\varepsilon}$. Since $a_n > 0 \; \forall n \in \mathbb{N}$, we conclude that $\sum a_n$ converges. □

Continuity

Definition. Let $f\colon I \to \mathbb{R}$ be a function, where $I \subset \mathbb{R}$. We say that f is *continuous at* $x_0 \in I$ if for every sequence $\{x_n\} \subset I$ with

$$\lim_{n\to\infty} x_n = x_0,$$

we have

$$\lim_{n\to\infty} f(x_n) = f(x_0).$$

If f is continuous at every point of a set $S \subset I$, then we say f is *continuous on* S. We say f is *continuous* if it is continuous on I.

Theorem. Let $I \subset \mathbb{R}$ and $f\colon I \to \mathbb{R}$ be a function. Then f is continuous at $x_0 \in I$ if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in I$ with $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| < \varepsilon.$$

Proof. ($\Rightarrow$) By contradiction: Assume $\exists\, \varepsilon_0 > 0$ such that $\forall\, \delta_0 > 0$, $\exists\, x_\delta \in I$ with $|x_\delta - x_0| < \delta_0$ but $|f(x_\delta) - f(x_0)| \geq \varepsilon_0$.

Take $\delta = \frac{1}{n}$. Then we get a sequence $\{x_n\} \subset I$ with $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \varepsilon_0$. Now, since $|x_n - x_0| < \frac{1}{n}$, we have $\lim_{n\to\infty} x_n = x_0$.

Hence, by continuity at x_0, we must have $\lim_{n\to\infty} f(x_n) = f(x_0)$, which contradicts $|f(x_n) - f(x_0)| \geq \varepsilon_0\ \forall\, n \in \mathbb{N}$.

($\Leftarrow$) Let $\{x_n\}$ be a sequence in I with $\lim_{n\to\infty} x_n = x_0$. We need to show $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Let $\varepsilon > 0$. Then $\exists\, \delta > 0$ such that $x \in I$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

Since $\lim_{n\to\infty} x_n = x_0$, $\exists\, n_0 \in \mathbb{N}$ such that $|x_n - x_0| < \delta$ for all $n \geq n_0$. Therefore, $|f(x_n) - f(x_0)| < \varepsilon$ for all $n \geq n_0$. Hence, $\lim_{n\to\infty} f(x_n) = f(x_0)$. ☐

Example.

1. $f(x) = x^n$, $n \geq 1$: It is easier to check with the ε-δ definition.
2. $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$: f is continuous at 0. Use the ε-δ method; $|100x| \leq x$.

Theorem. Let $f\colon I \to \mathbb{R}$ be continuous at $x_0 \in I$. Then so are $|f|$ and kf for any $k \in \mathbb{R}$.

Proof. For $|f|$, use

$$||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)|.$$

For kf, use

$$|kf(x) - kf(x_0)| = |k||f(x) - f(x_0)|.$$

☐

Theorem. Let f and g be continuous at $x_0 \in I$. Then:

1. $f + g$ is continuous at x_0.
2. fg is continuous at x_0.
3. If $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0.

Proof. Let $\{x_n\} \subset I$ be such that $x_n \to x_0$. Then:

$$\lim(f+g)(x_n) = \lim f(x_n) + \lim g(x_n) = f(x_0) + g(x_0).$$

$$\lim(fg)(x_n) = \lim f(x_n) \cdot \lim g(x_n) = f(x_0)g(x_0).$$

$$\lim \frac{f}{g}(x_n) = \frac{\lim f(x_n)}{\lim g(x_n)} = \frac{f(x_0)}{g(x_0)}.$$

Here we use the fact that $g(x) \neq 0$ for sufficiently large n as a consequence of the fact that $g(x_0) \neq 0$ and g is continuous at x_0. □

Theorem. If $f\colon I \to J$ is continuous at $x_0 \in I$ and $g\colon J \to K$ is continuous at $f(x_0) \in J$, then $g \circ f$ is continuous at x_0.

Proof. Let $\varepsilon > 0$. As g is continuous at $f(x_0)$, $\exists\, \delta > 0$ such that

$$y \in J \text{ and } |y - f(x_0)| < \delta \implies |g(y) - g(f(x_0))| < \varepsilon. \qquad (*)$$

Since f is continuous at x_0, $\exists\, \eta > 0$ such that

$$x \in I \text{ and } |x - x_0| < \eta \implies |f(x) - f(x_0)| < \delta.$$

Thus, for $x \in I$ with $|x - x_0| < \eta$, we have $|f(x) - f(x_0)| < \delta$, and by $(*)$,

$$|g(f(x)) - g(f(x_0))| < \varepsilon.$$

Hence, $g \circ f$ is continuous at x_0. □

Definition. We say that $f\colon I \to \mathbb{R}$ is bounded if $f(I) = \{f(x)\colon x \in I\}$ is bounded in $\mathbb{R}$, i.e., if $\exists\, M \geq 0$ such that $|f(x)| \leq M\ \forall x \in I$.

Theorem. Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is bounded. Moreover, f attains its supremum and infimum values in $[a,b]$, i.e., $\exists\, x_1, x_2 \in [a,b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a,b].$$

Proof. **"f is bounded"**: By contradiction, assume f is not bounded. Then for each $n \in \mathbb{N}$, $\exists\, x_n \in [a,b]$ such that $|f(x_n)| > n$, which implies $\lim|f(x_n)| = \infty$.

But $x_n \in [a,b] \implies \{x_n\}$ is bounded $\implies \{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ converging to $x_0 \in \mathbb{R}$. Since $a \leq x_{n_k} \leq b$, $x_0 \in [a,b]$.

By continuity of f on $[a,b]$, we have $f(x_{n_k}) \to f(x_0)$, and hence $|f(x_{n_k})| \to |f(x_0)|$. This contradicts $\lim|f(x_n)| = \infty$.

Now let $M = \sup\{f(x)\colon x \in [a,b]\}$. Then $M < \infty$ as f is bounded. For $n \in \mathbb{N}$, $\exists\, y_n \in [a,b]$ such that

$$M - \frac{1}{n} < f(y_n) \leq M.$$

Hence,

$$\lim f(y_n) = M.$$

Since $\{y_n\}$ is bounded, by Bolzano-Weierstrass, $\exists$ a convergent subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to $y_0 \in [a, b]$. Since f is continuous at y_0, we get

$$f(y_0) = \lim f(y_{n_k}) = M.$$

The argument is similar for the infimum. □ □

Remark. 1. $f(x) = x$ is continuous on $(0, 1)$ but does not attain its supremum or infimum.

2. $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$ but unbounded on $(0, 1)$.

Theorem (Intermediate Value Theorem)**.** Let I be an interval, and $f : I \to \mathbb{R}$ be continuous. Then f has the intermediate value property: if $a, b \in I$, $a < b$, and y lies between $f(a)$ and $f(b)$, then $\exists\, x \in (a, b)$ such that $f(x) = y$.

Proof. Without loss of generality, assume that $f(a) < f(b)$ (otherwise work with $-f$). Let $f(a) < y < f(b)$, and set

$$A = \{x \in [a, b] : f(x) < y\}.$$

Then $a \in A$, and A is bounded. Let $x_0 = \sup A$. We want to check that $f(x_0) = y$.

Claim 1: $f(x_0) \leq y$.
Since $x_0 = \sup A$, for $n \in \mathbb{N}$, $\exists\, x_n \in A$ such that $x_0 - \frac{1}{n} < x_n \leq x_0$. Thus, $\lim x_n = x_0 \implies \lim f(x_n) = f(x_0)$. Since $f(x_n) \leq y$ for all $n \in \mathbb{N}$, we have $f(x_0) \leq y$.

Claim 2: $y \leq f(x_0)$.
Let $a_n = \min\{x_0 + \frac{1}{n}, b\}$. Then $x_0 < a_n \leq x_0 + \frac{1}{n}$ for large n. By Claim 1, $\lim a_n = x_0 \implies \lim f(a_n) = f(x_0)$. Also, $f(a_n) > y$ for all $n \in \mathbb{N}$, so $f(x_0) \geq y$.

Combining Claim 1 and Claim 2, we conclude that $f(x_0) = y$. □

Corollary. Let I be an interval, and $f : I \to \mathbb{R}$ be continuous. Then $f(I)$ is also an interval (or a singleton).

Proof. As f is continuous, $J = f(I)$ has the property that if $y_1, y_2 \in J$ with $y_1 < y_2$, then $(y_1, y_2) \subseteq J$.
Case 1: $\sup J = \inf J \implies f(I)$ is a singleton set.
Case 2: $\sup J > \inf J$.
Let $\inf J < y < \sup J$. We want to show $y \in J$, which implies J is an interval with endpoints $\inf J$ and $\sup J$ (which may or may not be in J).
Since $\inf J < y$, $\exists\, y_1 \in J$ such that $\inf J \leq y_1 < y$. Similarly, since $\sup J > y$, $\exists\, y_2 \in J$ such that $y < y_2 \leq \sup J$.
Thus, $y_1 < y < y_2$, and $y_1, y_2 \in J$. By the intermediate value property, $y \in J$.
Therefore, J is an interval (or a singleton). □ □

Theorem. Let I be an interval and $f : I \to \mathbb{R}$ be strictly increasing such that $f(I)$ is an interval. Then f is continuous.

Proof. Let $x_0 \in I \setminus \{\inf I, \sup I\}$. As f is strictly increasing, $f(x)$ is not an endpoint of $f(I)$, which is an interval. So, $\exists\, \varepsilon > 0$ such that

$$[f(x_0) - \varepsilon, f(x_0) + \varepsilon] \subseteq f(I).$$

Let $\varepsilon > 0$. Since $[f(x_0) - \varepsilon, f(x_0) + \varepsilon] \subseteq f(I)$, $\exists\, x_1, x_2 \in I$ such that

$$f(x_1) = f(x_0) - \varepsilon \quad \text{and} \quad f(x_2) = f(x_0) + \varepsilon, \quad \text{with} \quad x_1 < x_0 < x_2.$$

For $x \in (x_1, x_2)$, we have $f(x_1) < f(x) < f(x_2) \implies |f(x) - f(x_0)| < \varepsilon$. Let $\delta = \min\{x_2 - x_0, x_0 - x_1\}$, and conclude that f is continuous.

Now assume $x_0 = \inf I > -\infty$. Then $f(x_0) = \inf f(I)$. Let $\varepsilon > 0$ small enough such that $[f(x_0), f(x_0) + \varepsilon] \subseteq f(I)$. Then for $z \in (0, \varepsilon)$, $\exists\, x \in I$ such that $f(x) = f(x_0) + z$.

As f is increasing, we have $x_0 < x_2$, and for all $x \in (x_0, x_2)$, we have

$$f(x_0) < f(x) < f(x_2) \implies |f(x) - f(x_0)| < \varepsilon.$$

Choose $\delta = |x_1 - x_2|$. □

Corollary. Let I be an interval, and $f \colon I \to \mathbb{R}$ be continuous and strictly increasing. Then $f(I)$ is an interval. Let $f^{-1} \colon f(I) \to I$ be the inverse of f. Then f^{-1} is continuous and strictly increasing.

Proof. Since f is continuous and strictly increasing on an interval, by a previous corollary, $f(I)$ is an interval. Since f is strictly increasing, it is one-to-one on I, and hence f^{-1} is well-defined.

In view of the previous theorem, it is enough to check that f^{-1} is strictly increasing.

Let $y_1, y_2 \in f(I)$ with $y_1 < y_2$. Then $\exists!$ (unique) $x_1, x_2 \in I$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since f is strictly increasing, we get $x_1 < x_2 \implies f^{-1}(y_1) < f^{-1}(y_2)$.

Thus, f^{-1} is strictly increasing. □

Theorem. Let f be injective and continuous on an interval I. Then f is strictly increasing or strictly decreasing.

Proof. **Claim 1:** Let $a, b, c \in I$ with $a < b < c$. Then $f(b)$ lies between $f(a)$ and $f(c)$.

Proof of Claim 1: Assume not, and let $\max\{f(a), f(c)\} < f(b)$. Let

$$\max\{f(a), f(c)\} < y < f(b).$$

Then by the intermediate value theorem, $\exists\, x_1 \in (a, b)$ and $x_2 \in (b, c)$ such that

$$f(x_1) = y \quad \text{and} \quad f(x_2) = y,$$

which is a contradiction to the fact that f is injective.

Let $a, b, c \in I$ and $f(a) < f(b)$. We show below that for any $x_1, x_2 \in I$ with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Claim 2: If $f(x) < f(a) < f(c)$ for $x \in (a, c)$, then:

1. $f(x) > f(a)$ for $x > a$.
2. $f(x) < f(c)$ for $x < c$.

Proof of Claim 2: Since $x < a < c$, we have $f(x)$ must lie between $f(a)$ and $f(c)$.

As $f(c) > f(a)$, we have $f(x) < f(c)$. For $x < c$, we have either:

$$x < a \implies f(x) \text{ lies between } f(a), f(c), \quad \text{or} \quad x > b \implies f(x) > f(a).$$

Combining these, f is strictly increasing or strictly decreasing.

Let $y_1, y_2 \in f(I)$, and suppose $y_1 < y_2$. Then:

1. If f is strictly increasing, $f^{-1}(y_1) < f^{-1}(y_2)$.
2. If f is strictly decreasing, $f^{-1}(y_1) > f^{-1}(y_2)$.

Thus, f must be monotone. □