Math 131B Running Notes

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1 Preliminaries

Definition (Metric)**.** Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a *metric* if it satisfies the following properties for all $x, y, z \in X$:

(i) $d(x, x) = 0$;

(ii) $d(x, y) > 0$ if $x \neq y$;

(iii) $d(x, y) = d(y, x)$;

(iv) $d(x, z) \leq d(x, y) + d(y, z) \quad$ (*triangle inequality*).

Definition (Metric Space)**.** The pair (X, d) is called a *metric space.*

Definition (Taxicab Metric)**.** The *taxicab metric* on $\mathbb{R}^n$ is defined as:

$$d_{\text{taxi}}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{n} |x_i - y_i|.$$

This is a metric.

Definition (Discrete Metric)**.** Let X be a non-empty set. The *discrete metric* on X is defined as:

$$d_{\text{disc}}(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{otherwise.} \end{cases}$$

Definition (Euclidean Metric on $\mathbb{R}$)**.** The *Euclidean metric* on $\mathbb{R}$ is defined as:

$$d(x, y) = |x - y|,$$

where $x, y \in \mathbb{R}$.

Definition (Euclidean Metric on $\mathbb{R}^n$)**.** The *Euclidean metric* (or ℓ^2-metric) on $\mathbb{R}^n$ is defined as:

$$d_{\text{Euclidean}}(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}},$$

where $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Definition (Convergence of a Sequence in a Metric Space)**.** Let (X, d) be a metric space. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X. We say that $(x_n)_{n=1}^{\infty}$ *converges* to $x_0 \in X$ (denoted $x_n \to x_0$) if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(x_n, x_0) < \epsilon.$$

Remark. The condition $d(x_n, x_0) < \epsilon$ is equivalent to $|d(x_n, x_0) - 0| < \epsilon$.

Thus, $x_n \to x_0$ if and only if

$$\lim_{n\to\infty} d(x_n, x_0) = 0.$$

Proposition. Let $(x_n)_{n=1}^{\infty}$ be a sequence in some discrete metric space (X, d_{disc}).

If $(x_n)_{n=1}^{\infty}$ converges, then the sequence is eventually constant.

Proposition. Let $(x^{(k)})_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$, where each $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$.

With the standard Euclidean metric, the sequence $(x^{(k)})_{k=1}^{\infty}$ converges to $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ if and only if each component sequence $(x_i^{(k)})_{k=1}^{\infty}$ converges to x_i in $\mathbb{R}$ for all $i = 1, 2, \ldots, n$.

Remark. If $(x_n)_{n=1}^{\infty}$ is eventually constant, then it converges in any metric space.

Proposition (Uniqueness of Limits)**.** Limits of sequences are unique.

Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space (X, d). Suppose $x_n \to x \in X$ and $x_n \to y \in X$. Then $x = y$.

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2 Point Set Topology

Definition. Let (X, d) be a metric space. Let $x_0 \in X$ and $r \in \mathbb{R}_{>0}$. We define the *ball* centered at x_0 with radius r as

$$B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}.$$

Definition. Let (X, d) be a metric space. Let $U \subseteq X$. We say that U is *open* if

$$\forall x \in U, \exists r > 0 \text{ such that } B(x, r) \subseteq U,$$

where $B(x, r)$ denotes the ball centered at x with radius r.

Definition. Let (X, d) be a metric space, and let $E \subseteq X$.

(i) A point $x_0 \in X$ is an *interior point* of E if $\exists r > 0$ such that $B(x_0, r) \subseteq E$.

(ii) A point $x_0 \in X$ is an *exterior point* of E if $\exists r > 0$ such that $B(x_0, r) \cap E = \emptyset$.

(iii) A point $x_0 \in X$ is a *boundary point* of E if it is neither an interior point nor an exterior point of E.

(iv) A point $x_0 \in X$ is an *adherent point* of E if $\forall r > 0$, $B(x_0, r) \cap E \neq \emptyset$.

Let $E \subseteq X$ be a subset of a metric space (X, d). We use the following **notations:**

- $\text{int}(E) := \{\text{interior points of } E\}$
- $\text{ext}(E) := \{\text{exterior points of } E\}$
- $\partial E := \{\text{boundary points of } E\}$
- $\overline{E} := \{\text{adherent points of } E\}$ (closure of E).

Definition. Let (X, d) be a metric space, and let $E \subseteq X$. We say that E is **closed** if it contains all of its adherent points, i.e.,

$$\overline{E} \subseteq E.$$

Remark. Let $E \subseteq X$, where (X, d) is a metric space. The following hold:

(i) $\text{int}(E) \subseteq E$, with equality if and only if E is open.

(ii) $E \subseteq \overline{E}$, with equality if and only if E is closed.

(iii) $\text{ext}(E) \cap E = \emptyset$, where $\text{ext}(E)$ denotes the set of exterior points of E.

(iv) E is closed if and only if $\overline{E} \subseteq E$.

Proposition. Let (X, d) be a metric space. Let $x_0 \in X$ and $R > 0$. Then, the ball $B(x_0, R)$ is open.

Fact. Let (X, d) be a metric space.

(i) $\emptyset$ is open and closed.

(ii) X is open and closed.

(iii) If $\{U_i\}_{i \in I}$ is a collection of open sets, then $\bigcup_{i \in I} U_i$ is open. (Countable?)

(iv) If $\{F_i\}_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} F_i$ is closed.

(v) If U, V are open, then $U \cup V$ is open.

  - By induction, finite unions of open sets are open.

(vi) If E, F are closed, then their finite union $E \cup F$ is closed.

(vii) $\text{int}(E)$ is always open.

(viii) $\overline{E}$ is always closed.

Definition (Subspaces (1.3))**.** If (X, d) is a metric space and $Y \subseteq X$, then $(Y, d|_Y)$ is a metric space as well, obtained by restricting d to points of Y.

Definition (Relative Openness and Closedness)**.** Let (X, d) be a metric space, $Y \subseteq X$, and $E \subseteq Y$.

- E is called *relatively open* in Y if E is open in (Y, d).
- E is called *relatively closed* in Y if E is closed in (Y, d).

Definition (Subballs)**.**

$$B_X(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$$
$$B_Y(y_0, r) := \{y \in Y \mid d(y_0, y) < r\}$$

Since $Y \subseteq X$, it follows that:

$$B_Y(y_0, r) = B_X(x_0, r) \cap Y.$$

Not formal name for these balls

Example. Consider $(\mathbb{R}, d_{\text{std}})$ and $Y = [0, \infty)$. We claim $[0, 1)$ is relatively open in Y. Another way to see this is to note that $[0, 1) = B_Y(0, 1)$, and we have shown that all balls are open.

Proposition. Let (X, d) be a metric space, $Y \subseteq X$, and $E \subseteq Y$. Then:

(i) E is relatively open in Y if and only if there exists $E' \subseteq X$ open such that $E = E' \cap Y$.

(ii) E is relatively closed in Y if and only if there exists $E' \subseteq X$ closed such that $E = E' \cap Y$.

Definition (Cauchy Sequence)**.** Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is called a *Cauchy sequence* if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ such that } n, m \geq N \implies d(x_n, x_m) < \epsilon.$$

Proposition. Let (X, d) be a metric space, and $E \subseteq X$. Then E is closed if and only if $X \setminus E$ is open.

Proof. **(⇒)** Suppose E is closed. Then $\overline{E} = E$.

We want to show that if $x_0 \in X \setminus E$, then $x_0 \in \text{int}(X \setminus E)$. Let $x_0 \in X \setminus E$. Since $E = \overline{E}$, $x_0 \notin \overline{E}$. This means x_0 is not adjacent to E, so x_0 is in the exterior of E. Therefore, there exists $r_0 > 0$ such that $B(x_0, r) \cap E = \emptyset$. This implies $B(x_0, r) \subseteq X \setminus E$, so $x_0 \in \text{int}(X \setminus E)$.

(⇐) Assume $X \setminus E$ is open. Thus, $X \setminus E = \text{int}(X \setminus E)$.

We want to show that $x_0 \in \overline{E} \implies x_0 \in E$. Let $x_0 \in \overline{E}$. Then, for all $r > 0$, $B(x_0, r) \cap E \neq \emptyset$. Equivalently, for all $r > 0$, $B(x_0, r) \nsubseteq X \setminus E$. Hence, $x_0 \notin \text{int}(X \setminus E)$, which implies $x_0 \notin X \setminus E$. Therefore, $x_0 \in E$, and so $\overline{E} \subseteq E$. Thus, E is closed. □

Example (Open/Closed Discrete Metric Space)**.** Let (X, d_{disc}) be a discrete metric space, and let $E \subseteq X$. Then:

Claim: E is open.

Proof. Let $x_0 \in E$. Choose $r = 1$. Then,

$$B(x_0, 1) = \{x \in X \mid d_{\text{disc}}(x, x_0) < 1\} = \{x_0\} \subseteq E.$$

Therefore, E is open. □

$$\implies \text{Every subset of a discrete metric space is open!}$$

In particular, $X \setminus E$ is open, which implies E is closed.

$$\implies \text{Every subset of a discrete metric space is both open and closed.}$$

Definition (Subsequences)**.** Let $(x_n)_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d). Suppose that $n_1, n_2, n_3, \dots$ is an increasing sequence of integers such that $n_j \geq m$ for all j, satisfying:

$$m \leq n_1 < n_2 < n_3 < \cdots .$$

Then the sequence $(x_{n_j})_{j=1}^{\infty}$ is called a *subsequence* of the original sequence $(x_n)_{n=m}^{\infty}$.

Theorem. Let $(x_n)_{n=m}^{\infty}$ be a sequence in a metric space (X, d) that converges to some limit x_0. Then every subsequence $(x_{n_j})_{j=1}^{\infty}$ of that sequence also converges to x_0.

Definition (Limit Points)**.** Suppose that $(x_n)_{n=m}^{\infty}$ is a sequence of points in a metric space (X, d), and let $L \in X$. We say that L is a *limit point* of $(x_n)_{n=m}^{\infty}$ if and only if for every $N \geq m$ and $\epsilon > 0$, there exists an $n \geq N$ such that $d(x_n, L) \leq \epsilon$.

Definition (Cauchy Sequence)**.** Let $(x_n)_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d). We say that this sequence is a *Cauchy sequence* if and only if for every $\epsilon > 0$, there exists an $N \geq m$ such that $d(x_j, x_k) < \epsilon$ for all $j, k \geq N$.

Lemma (Convergent Sequences are Cauchy Sequences)**.** Let $(x_n)_{n=m}^{\infty}$ be a sequence in (X, d) which converges to some limit x_0. Then $(x_n)_{n=m}^{\infty}$ is also a Cauchy sequence.

Definition (Complete Metric Space)**.** A metric space (X, d) is said to be *complete* if and only if every Cauchy sequence in (X, d) is convergent in (X, d).

Proposition (1.4.12)**.** This proposition states that

(a) Let (X, d) be a metric space, and let $(Y, d|_{Y \times Y})$ be a subspace of (X, d). If $(Y, d|_{Y \times Y})$ is complete, then Y must be closed in X.

(b) Conversely, suppose that (X, d) is a complete metric space, and Y is a closed subset of X. Then the subspace $(Y, d|_{Y \times Y})$ is also complete.

Definition (Open Cover)**.** Let (X, d) be a metric space and $E \subseteq X$. A collection $\{U_i\}_{i \in I}$ of open sets is called an *open cover* of E if

$$E \subseteq \bigcup_{i \in I} U_i.$$

Definition (Compact Set)**.** Let (X, d) be a metric space, and let $E \subseteq X$. Then, E is said to be *compact* if every open cover of E admits a finite subcover. That is, for every collection of open sets $\{U_i\}_{i \in I}$ such that

$$E \subseteq \bigcup_{i \in I} U_i,$$

there exists a finite subset $\{i_1, \ldots, i_n\} \subseteq I$ such that

$$E \subseteq \bigcup_{j=1}^{n} U_{i_j}.$$

Definition (Bounded Set)**.** Let (X, d) be a metric space, and let $E \subseteq X$. We say that E is *bounded* if there exists a point $x_0 \in X$ and a real number $R > 0$ such that

$$E \subseteq B(x_0, R),$$

where $B(x_0, R) = \{x \in X : d(x, x_0) < R\}$ is the open ball of radius R centered at x_0.

Remark. In fact, finite sets are always compact in any metric space (X, d).

Remark (Differences between Limit Points and Adherent Points)**.** The key differences between limit points and adherent points are:

1. **Includes the Point:** A limit point does not include the point itself unless it is approached by other points in the set. An adherent point always includes the point if it belongs to the set.
2. **Neighborhood Condition:** A limit point requires every neighborhood to contain another distinct point of the set. An adherent point requires every neighborhood to contain at least one point of the set, including itself.
3. **Relation to Closure:** All limit points are in the closure, but adherent points form the entire closure, including the set itself.

Proposition. Let (X, d) be a metric space and $E \subseteq X$. If E is compact, then E must be closed and bounded.

Proof. We first show that E is bounded. Suppose E is compact. Pick any $x_0 \in X$. Note that

$$E \subseteq \bigcup_{n \in \mathbb{N}} B(x_0, n),$$

where $B(x_0, n)$ denotes the open ball of radius n centered at x_0. The collection of all such balls forms an open cover of E.

By the compactness of E, there exist finitely many indices $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$E \subseteq \bigcup_{j=1}^{k} B(x_0, n_j).$$

Let $N = \max\{n_1, \ldots, n_k\}$. Then $E \subseteq B(x_0, N)$, so E is bounded.

Hence, E is bounded as desired. □

Definition (Sequential Compactness)**.** Let (X, d) be a metric space and $E \subseteq X$. We say E is **sequentially compact** if every sequence $(x_n)_{n=1}^{\infty} \subseteq E$ has a subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to some $x \in E$.

Theorem (Bolzano-Weierstrass)**.** If $(x_n)_{n=1}^{\infty}$ is a bounded sequence in $(\mathbb{R}, d_{\text{std}})$, then there exists a subsequence that converges to some real number.

Theorem (Heine-Borel)**.** Let $E \subseteq (\mathbb{R}^n, d_{\text{std}})$. If E is closed and bounded, then E is sequentially compact.

Proposition. Let (X, d) be a metric space and $E \subseteq X$. If E is compact, then E must be closed and bounded.

Proof. We first show that E is bounded. Suppose E is compact. Pick any $x_0 \in X$. Note that

$$E \subseteq \bigcup_{n \in \mathbb{N}} B(x_0, n),$$

where $B(x_0, n)$ denotes the open ball of radius n centered at x_0. The collection of all such balls forms an open cover of E.

By the compactness of E, there exist finitely many indices $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$E \subseteq \bigcup_{j=1}^{k} B(x_0, n_j).$$

Let $N = \max\{n_1, \ldots, n_k\}$. Then $E \subseteq B(x_0, N)$, so E is bounded.

Hence, E is bounded as desired.

Suppose E is compact in (X, d) and let $x \in X \setminus E$. We claim x is not an adherent point of E. Equivalently, if x *were* an adherent point, then every ball $B(x, r)$ meets E.

Construct the open cover

$$\{\, X \setminus \{x\}\} \;\cup\; \{B(e, \tfrac{1}{n}) : e \in E,\, n \in \mathbb{N}\}$$

of X. Since E is compact, it suffices to show E must lie entirely in a finite subcover disjoint from x. By refining or adjusting the open sets, one shows no cluster can form at x unless $x \in E$. (Alternatively, a standard argument uses sequences or adherent points: in a metric space, E is closed iff every convergent sequence in E has its limit in E. If $x \notin E$ were a limit of points of E, that would contradict the compactness or lead to $x \in E$.) Thus E contains all its limit (adherent) points, i.e. E is closed. $\square$

Definition (Sequential Compactness)**.** Let (X, d) be a metric space and $E \subseteq X$. We say E is **sequentially compact** if every sequence $(x_n)_{n=1}^{\infty} \subseteq E$ has a subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to some $x \in E$.

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3 Continuity

Definition (Continuity in Metric Spaces)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is *continuous at* $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon.$$

Proposition. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous if for every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.

Proposition. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. If f is continuous at $x_0 \in X$, and g is continuous at $f(x_0)$, then the composition $g \circ f : X \to Z$ is continuous at x_0.

Theorem. Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space. Let $f : X \to Y$ be a function. Then the following statements are equivalent:

(a) f is continuous.

(b) Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X that converges to some point $x_0 \in X$ with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.

(c) Whenever V is an open set in Y, the set $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open set in X.

(d) Whenever F is a closed set in Y, the set $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is a closed set in X.

Corollary (Continuity Preserved by Composition)**.** Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces.

(a) If $f : X \to Y$ is continuous at a point $x_0 \in X$, and $g : Y \to Z$ is continuous at $f(x_0)$, then the composition $g \circ f : X \to Z$, defined by $(g \circ f)(x) := g(f(x))$, is continuous at x_0.

(b) If $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is also continuous.

Corollary. Let (X, d) be a metric space, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions. Let c be a real number.

(a) If $x_0 \in X$ and f and g are continuous at x_0, then the functions

$$f + g : X \to \mathbb{R}, \quad f - g : X \to \mathbb{R}, \quad \max(f, g) : X \to \mathbb{R}, \quad \min(f, g) : X \to \mathbb{R}, \quad \text{and } cf : X \to \mathbb{R}$$

are also continuous at x_0. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \mathbb{R}$ is also continuous at x_0.

(b) If f and g are continuous, then the functions

$$f + g : X \to \mathbb{R}, \quad f - g : X \to \mathbb{R}, \quad \max(f, g) : X \to \mathbb{R}, \quad \min(f, g) : X \to \mathbb{R}, \quad \text{and } cf : X \to \mathbb{R}$$

are also continuous on X. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \mathbb{R}$ is also continuous on X.

Example.

- We know that $f(x) = x$ is continuous. This implies that all polynomials are continuous.
- We also know that $f(x, y) = x$ and $g(x, y) = y$ are continuous. Thus, all multivariate polynomials, such as $x^2y + 2y^3$, are continuous.

Theorem (Continuous Maps Preserve Compactness)**.** Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let $K \subseteq X$ be any compact subset of X. Then the image

$$f(K) := \{f(x) : x \in K\}$$

of K is also compact.

Proposition (Maximum Principle)**.** Let (X, d) be a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Then f is bounded. Furthermore, if X is non-empty, then f attains its maximum at some point $x_{\max} \in X$ and also attains its minimum at some point $x_{\min} \in X$.

Definition (Uniform Continuity)**.** Let $f : X \to Y$ be a map from one metric space (X, d_X) to another (Y, d_Y). We say that f is *uniformly continuous* if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$d_Y(f(x), f(x')) < \epsilon \quad \text{whenever } x, x' \in X \text{ and } d_X(x, x') < \delta.$$

Every uniformly continuous function is continuous, but not conversely. However, if the domain X is compact, then the two notions are equivalent.

Theorem. Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose that (X, d_X) is compact. If $f : X \to Y$ is a function, then f is continuous if and only if it is uniformly continuous.

Definition (Disconnected and Connected Sets)**.** Let (X, d) be a metric space, and let $E \subseteq X$.

- E is *disconnected* if there exist $U, V \subseteq E$, non-empty, relatively open subsets (with respect to E), such that:
$$E = U \cup V \quad \text{and} \quad U \cap V = \emptyset.$$
- E is *connected* if it is not disconnected.

Example.

- Consider $(\mathbb{R}, d_{\text{std}})$ and let $E = \{0, 1\}$. Then E is disconnected. Define $U = \{0\}$ and $V = \{1\}$. We have:
$$E = U \cup V \quad \text{and} \quad U \cap V = \emptyset.$$
- Let $F = \mathbb{R} \setminus \{0\}$. Then F is also disconnected. Define $U = (-\infty, 0)$ and $V = (0, \infty)$, which are open in $\mathbb{R}$ and thus relatively open in F. Both U and V are non-empty, and:
$$F = U \cup V \quad \text{and} \quad U \cap V = \emptyset.$$

Theorem. Let X be a non-empty subset of the real line $\mathbb{R}$. Then the following statements are equivalent:

(a) X is connected.

(b) Whenever $x, y \in X$ and $x < y$, the interval $[x, y]$ is also contained in X.

(c) X is an interval (in the sense of Definition 9.1.1).

Theorem (Continuity Preserves Connectedness)**.** Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let E be any connected subset of X. Then $f(E)$ is also connected.

Corollary (Intermediate Value Theorem)**.** Let $f : X \to \mathbb{R}$ be a continuous map from one metric space (X, d_X) to the real line. Let E be any connected subset of X, and let a, b be any two elements of E. Let y be a real number between $f(a)$ and $f(b)$, i.e., either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists $c \in E$ such that $f(c) = y$.

Definition. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose for each $n \in \mathbb{N}$, we have a function $f_n : X \to Y$. We say that the sequence $(f_n)_{n=1}^{\infty}$ **converges pointwise** to some function $f : X \to Y$ if

$$\forall x \in X, \quad f_n(x) \to f(x) \quad (\text{in } Y).$$

Definition. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of functions. We say that $(f_n)_{n=1}^{\infty}$ **converges uniformly** to some function $f : X \to Y$ if

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \text{ such that } \forall x \in X, \forall n \geq N, \quad d_Y(f_n(x), f(x)) < \varepsilon.$$

Theorem. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f_n : X \to Y$ be a sequence of functions that converge uniformly to some function $f : X \to Y$.

If each f_n is continuous at $x_0 \in X$, then f will also be continuous at x_0.

Remark. If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

Proposition. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f_n : X \to Y$ be a sequence of functions that converge uniformly to some function $f : X \to Y$.

If each f_n is bounded, then f will also be bounded.

Remark. A function f is bounded if and only if $f(x)$ is bounded.

Definition. The ℓ^{∞} metric (sup-norm metric) on $\mathcal{B}(X \to Y)$ is defined as

$$d_{\infty}(f, g) := \sup_{x \in X} d_Y(f(x), g(x)).$$

Proposition. This is a well-defined metric on $\mathcal{B}(X \to Y)$.

Proposition. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $(\mathcal{B}(X \to Y), d_{\infty})$. Then,

$$f_n \to f \text{ with respect to } d_{\infty} \iff (f_n) \text{ converges uniformly.}$$

Fact. The space of continuous functions $\mathcal{C}(X \to Y)$ is a closed subset of $(\mathcal{B}(X \to Y), d_{\infty})$.

That is, if $f_n \to f$ in d_{∞} and $f_n \in \mathcal{C}(X \to Y)$ for all n, then $f \in \mathcal{C}(X \to Y)$. Hence, $\mathcal{C}(X \to Y)$ is closed in $\mathcal{B}(X \to Y)$.

Theorem. Let (X, d_X) and (Y, d_Y) be metric spaces. If (Y, d_Y) is complete, then the space of bounded functions $(\mathcal{B}(X \to Y), d_{\infty})$ is also complete.

Definition. Let $f_n : (X, d) \to (\mathbb{R}, d_{\text{std}})$ be a sequence of $\mathbb{R}$-valued functions. The symbol

$$\sum_{n=1}^{\infty} f_n$$

denotes the corresponding series. This represents a sequence of partial sums, where the N-th partial sum is given by

$$S_N = \sum_{n=1}^{N} f_n.$$

Each partial sum S_N is also an $\mathbb{R}$-valued function.

Definition (Convergence of series of functions)**.** Let $f_n : (X, d) \to (\mathbb{R}, d_{\text{std}})$ be a sequence of functions. Define the function f by

$$f = \sum_{n=1}^{\infty} f_n.$$

We say that the series $\sum_{n=1}^{\infty} f_n$:

- *Converges pointwise* to f if the sequence of partial sums

$$P_N = \sum_{n=1}^{N} f_n$$

  converges pointwise to f.

- *Converges uniformly* to f if the sequence of partial sums $(P_N)_{N=1}^{\infty}$ converges uniformly to f.

In either case, we may write

$$f = \sum_{n=1}^{\infty} f_n.$$

Example. Consider the sequence of functions $f_n : (-1, 1) \to \mathbb{R}$ defined by:

$$f_n(x) = x^n, \quad (n \geq 0).$$

The corresponding partial sum of the geometric series is given by:

$$P_n(x) = 1 + x + x^2 + \cdots + x^n.$$

Using the formula for the sum of a geometric series, we obtain:

$$P_n(x) = \frac{1 - x^{n+1}}{1 - x}, \quad x \neq 1.$$

As $n \to \infty$, we observe that:

$$P_n(x) \to \frac{1}{1 - x}, \quad \text{for } |x| < 1.$$

Hence, the infinite geometric series:

$$\sum_{n=0}^{\infty} x^n$$

converges pointwise to:

$$\frac{1}{1 - x}, \quad \text{for } |x| < 1.$$

Definition (Supremum Norm)**.** Let $f : (X, d_X) \to (Y, d_Y)$ be a bounded function. The *supremum norm* (or sup-norm) of f is defined as:

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

Remark. The supremum norm induces a metric on the space of bounded functions $B(X \to Y)$, given by:

$$d_\infty(f, g) = \|f - g\|_\infty, \quad \text{for } f, g \in B(X \to Y).$$

Theorem (Weierstrass M-test)**.** Let (X, d) be a metric space, and let $(f_n)_{n\geq 1}$ be a sequence in $C(X \to \mathbb{R})$. If the series

$$\sum_{n=1}^{\infty} \|f_n\|_\infty$$

converges, then the series

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly to some function $f \in C(X \to \mathbb{R})$.

Proof. Define $M_n = \|f_n\|_\infty$. Then for all n and for all $x \in X$, we have:

$$|f_n(x)| \leq M_n.$$

Consider the partial sums $P_n = \sum_{i=1}^{n} f_i$. For $n \geq m$, we estimate the sup-norm distance:

$$d_\infty(P_n, P_m) = \|P_n - P_m\|_\infty.$$

Expanding the definition,

$$\|P_n - P_m\|_\infty = \left\| \sum_{i=m+1}^{n} f_i \right\|_\infty .$$

By the triangle inequality,

$$\left| \sum_{i=m+1}^{n} f_i(x) \right| \leq \sum_{i=m+1}^{n} |f_i(x)| \leq \sum_{i=m+1}^{n} M_i.$$

Thus, taking the supremum over $x \in X$,

$$\left\| \sum_{i=m+1}^{n} f_i \right\|_\infty \leq \sum_{i=m+1}^{n} M_i.$$

Since $\sum M_i$ converges, for any $\varepsilon > 0$, there exists an index N such that for all $n, m \geq N$,

$$\sum_{i=m+1}^{n} M_i < \varepsilon.$$

This shows that $(P_n)_{n\geq 1}$ is Cauchy in $C(X \to \mathbb{R})$.

By a previous theorem, since $(\mathbb{R}, d_{\text{sup}})$ is complete, the space $C(X \to \mathbb{R})$ is also complete. Therefore, there exists a function $f \in C(X \to \mathbb{R})$ such that:

$$P_n \to f \quad \text{uniformly.}$$

□

4 Uniform Convergence and Integrals and Differentiation

Definition (Integrability via Upper and Lower Integrals)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We define the *lower integral* and *upper integral* as:

$$\mathcal{L}(f) = \sup \int_{[a,b]} f, \quad \mathcal{U}(f) = \inf \int_{[a,b]} f.$$

The function f is said to be *integrable* if:

$$\mathcal{L}(f) = \mathcal{U}(f).$$

In this case, the integral of f is given by:

$$\int_a^b f = \int_{[a,b]} f = \mathcal{U}(f) = \mathcal{L}(f).$$

Theorem. Let $(f_n)_{n\geq 1}$ be a sequence of integrable functions $f_n : [a, b] \to \mathbb{R}$. If $f_n \to f$ *uniformly* for some real-valued function f, then f is also integrable, and moreover,

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Definition (Continuously Differentiable Function)**.** Let $f : [a, b] \to \mathbb{R}$. We say that f is *continuously differentiable* (or C^1) if it is differentiable and its derivative f' is continuous. That is, $f \in C^1([a, b] \to \mathbb{R})$ if:

$$f' \in C([a, b] \to \mathbb{R}).$$

Theorem. Let $(f_n)_{n\geq 1}$ be a sequence of C^1 functions on $[a, b]$. Suppose that f_n' converges uniformly to some function $g : [a, b] \to \mathbb{R}$. Further, suppose that there exists some $x_0 \in [a, b]$ such that:

$$\lim_{n\to\infty} f_n(x_0) \quad \text{exists.}$$

Then, the sequence $(f_n)_{n\geq 1}$ converges uniformly to a differentiable function f, and:

$$f' = g.$$

Informally, this result states that if the derivatives f_n' converge uniformly, and the sequence $f_n(x_0)$ converges for some x_0, then f_n itself converges uniformly, and:

$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x).$$

Question: If $f_n \to f$ uniformly, does it follow that $f_n' \to f'$ (assuming f_n is differentiable and f is differentiable)?

Answer: Not necessarily.

Example. Consider the sequence of functions:

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}.$$

Each f_n is differentiable, and we observe that:

$$f_n(x) \to |x| \quad \text{uniformly}.$$

However, $|x|$ is not differentiable at $x = 0$. Thus, uniform convergence does not imply convergence of derivatives.

Example. Consider the sequence:

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

We observe that:

$$f_n(x) \to 0 \quad \text{uniformly}.$$

However, differentiating f_n, we get:

$$f_n'(x) = \sqrt{n}\cos(nx).$$

Evaluating at $x = 0$, we see:

$$f_n'(0) = \sqrt{n} \not\to 0.$$

Thus, even if $f_n \to f$ uniformly, the derivatives f_n' may not converge to f'.

Theorem (Theorem 3.6.1)**.** Let $[a, b]$ be an interval, and for each integer $n \geq 1$, let $f_n : [a, b] \to \mathbb{R}$ be a Riemann-integrable function. Suppose f_n converges uniformly on $[a, b]$ to a function $f : [a, b] \to \mathbb{R}$. Then f is also Riemann-integrable, and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Corollary (Corollary 3.6.2)**.** Let $[a, b]$ be an interval, and let $(f_n)_{n=1}^{\infty}$ be a sequence of uniformly Riemann-integrable functions on $[a, b]$ such that the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent. Then,

$$\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n.$$

Corollary (Corollary 3.7.3)**.** Let $[a, b]$ be an interval, and for every integer $n \geq 1$, let $f_n : [a, b] \to \mathbb{R}$ be a differentiable function whose derivative $f_n' : [a, b] \to \mathbb{R}$ is continuous. Suppose that the series

$$\sum_{n=1}^{\infty} \|f_n'\|$$

is absolutely convergent, where

$$\|f_n'\| := \sup_{x\in[a,b]} |f_n'(x)|$$

is the sup-norm of f_n', as defined in Definition 3.5.5. Suppose also that the series

$$\sum_{n=1}^{\infty} f_n(x_0)$$

is convergent for some $x_0 \in [a, b]$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$ to a differentiable function, and in fact,

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x).$$

5 Power Series

Definition (Power Series)**.** Let $a \in \mathbb{R}$. A *power series* centered at a is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n,$$

where $c_n \in \mathbb{R}$ are the coefficients of the series.

Example. The following is an example of a power series:

$$\sum_{n=0}^{\infty} n!(x-2)^n.$$

Example ((Non-example))**.** The following is **not** a power series:

$$\sum_{n=0}^{\infty} 2^x (x-3)^n.$$

This is because the coefficients $c_n = 2^x$ depend on x, violating the definition of a power series.

Definition (Radius of Convergence)**.** Let

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

be a power series. Define:

$$\rho = \limsup_{n\to\infty} |c_n|^{1/n}.$$

The *radius of convergence* of this series is given by:

$$R = \begin{cases} 0 & \text{if } \rho = +\infty, \\ +\infty & \text{if } \rho = 0, \\ \frac{1}{\rho} & \text{otherwise.} \end{cases}$$

Theorem (Root Test)**.** Let $\sum a_n$ be a series of real numbers. Define:

$$\alpha = \limsup |a_n|^{1/n}.$$

Then:

- If $\alpha < 1$, then $\sum a_n$ converges absolutely.
- If $\alpha > 1$, then $\sum a_n$ diverges.
- If $\alpha = 1$, the test is inconclusive.

Theorem. Let $\sum c_n(x-a)^n$ be a power series with radius of convergence R. Let $x_0 \in \mathbb{R}$. Then:

1. If $|x_0 - a| > R$, then $\sum c_n(x_0-a)^n$ diverges.
2. If $|x_0 - a| < R$, then $\sum c_n(x_0-a)^n$ converges absolutely.

Moreover, if $R > 0$ and $0 < r < R$, then:

1. $\sum c_n(x-a)^n$ converges *uniformly* on $[a-r, a+r]$, so that $\sum c_n(x-a)^n$ is continuous on $(a-R, a+R)$.
2. $\sum c_n(x-a)^n$ is differentiable on $(a-R, a+R)$ with derivative $\sum nc_n(x-a)^{n-1}$, which also converges uniformly on $[a-r, a+r]$.
3. If $a - R \leq y < z \leq a + R$, then:

$$\int_y^z \sum c_n(x-a)^n = \sum c_n \frac{(z-a)^{n+1} - (y-a)^{n+1}}{n+1}.$$

Definition (Real Analytic Function)**.** Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. We say that f is *real analytic* or *analytic* at $a \in E$ if there exists some $r > 0$ and a sequence of real coefficients $\{c_n\} \subset \mathbb{R}$ such that:

$$(a-r, a+r) \subseteq E$$

and for all $x \in (a-r, a+r)$, we have:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n.$$

Definition (Smooth Function)**.** Let $f : [a,b] \to \mathbb{R}$. We say that f is *smooth* (infinitely differentiable, or C^∞) if the k-th derivative exists for all $k \in \mathbb{N}$. That is,

$$\frac{d^k f}{dx^k} \text{ exists for all } k \in \mathbb{N}.$$

In other words, we write:

$$f \in C^\infty([a,b] \to \mathbb{R}).$$

Corollary. If $f : E \to \mathbb{R}$ is analytic, then f is smooth and all of its derivatives

$$\frac{d^k f}{dx^k}$$

are analytic as well.

Corollary (Taylor's Formula)**.** Let $f : E \to \mathbb{R}$ be analytic at $a \in E$, so that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

on $(a-r, a+r)$. Then, for every $k \in \mathbb{N}$,

$$f^{(k)}(a) = \frac{d^k f}{dx^k}(a) = k! \cdot c_k.$$

In particular, on $(a-r, a+r)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Corollary. Suppose $f : E \to \mathbb{R}$ is analytic at $a \in E$ and satisfies

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} d_n (x-a)^n$$

on some interval $(a-r, a+r)$. Then, we must have

$$c_n = d_n, \quad \forall n.$$

Proof. By Taylor's theorem,

$$c_n = \frac{f^{(n)}(a)}{n!} = d_n.$$

□

Fact. If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

on $(a-r, a+r)$, then f is analytic at every $x_0 \in (a-r, a+r)$.

Fact. If f and g are analytic at a, then $f+g$ and $f-g$ are also analytic at a.

Definition (Formal Product)**.** Let $(c_n)_{n=0}^{\infty}$ and $(d_n)_{n=0}^{\infty}$ be sequences of real numbers. The *formal product* of $\sum c_n$ and $\sum d_n$ is given by

$$\left(\sum_{n=0}^{\infty} c_n\right) \cdot \left(\sum_{n=0}^{\infty} d_n\right) = \sum_{n=0}^{\infty} e_n$$

where

$$e_n = \sum_{k=0}^{n} c_k \cdot d_{n-k}.$$

6 Exponential, Log, and Trigonometric Functions

Definition (Exponential Function)**.** For each $x \in \mathbb{R}$, we define

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Check: Convergence for all $x \in \mathbb{R}$.

Theorem (Properties of the Exponential Function)**.**

1. The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has an infinite radius of convergence and thus converges absolutely for all $x \in \mathbb{R}$. Hence, exp is analytic on $\mathbb{R}$.
2. exp is differentiable and
$$\exp'(x) = \exp(x).$$
3. exp is continuous and for any $a < b \in \mathbb{R}$, exp is integrable on $[a, b]$, with
$$\int_a^b \exp(t)\, dt = \exp(b) - \exp(a).$$
4. For all $x, y \in \mathbb{R}$,
$$\exp(x + y) = \exp(x) \exp(y).$$
5. $\exp(0) = 1$ and for all x,
$$\exp(x) > 0 \quad \text{and} \quad \exp(-x) = \frac{1}{\exp(x)}.$$
6. $\exp(x)$ is strictly increasing, i.e.,
$$x < y \quad \Rightarrow \quad \exp(x) < \exp(y).$$

Definition (Exponential function)**.** Define the exponential function as

$$e := \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Proposition. If $x \in \mathbb{Q}$, then $\exp(x) = e^x$.

Proof. We prove the result in three steps.

1. **Proof for integer exponents by induction:**
   We prove by induction on p that $\exp(p) = e^p$ for $p \in \mathbb{Z}_{\geq 0}$.
   Base case: For $p = 1$,
   $$\exp(1) = e = e^1$$
   by definition.
   Inductive step: Assume $\exp(p) = e^p$ for some $p \geq 1$. Then,
   $$\exp(p + 1) = \exp(p) \exp(1) = e^p e = e^{p+1}.$$
   Thus, the result holds for all $p \geq 0$.

2. **Proof for all integers:**
   Suppose $p \in \mathbb{Z}$. We already know that $\exp(p) = e^p$ for $p \geq 0$.
   If $p < 0$, we use the functional equation $\exp(-p)\exp(p) = 1$, which implies
   $$\exp(p) = \frac{1}{\exp(-p)} = \frac{1}{e^{-p}} = e^p.$$

3. **Proof for rational exponents:**
   Suppose $x = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then,
   $$p = x \cdot q.$$
   Applying the functional equation,
   $$\exp(p) = \exp(q \cdot x) = \exp(x + x + \cdots + x) \quad (q \text{ times})$$
   $$= \exp(x)\exp(x)\ldots\exp(x) = (\exp(x))^q.$$
   Since we know $\exp(p) = e^p$, we obtain
   $$e^p = (\exp(x))^q.$$
   Taking the qth root on both sides,
   $$e^{p/q} = \exp(x).$$
   Thus, the result holds for all $x \in \mathbb{Q}$.

□

Proposition. For all $x \in \mathbb{R}$, we have
$$\exp(x) = e^x.$$

Lemma. Let $n \in \mathbb{N}$. Then, for all $y \in \left[\frac{1}{2^n}, 2^n\right]$, there exists $x \in \mathbb{R}$ such that
$$\exp(x) = y.$$

Proposition. The image of the exponential function is
$$\exp(\mathbb{R}) = (0, \infty).$$

Corollary. The exponential function
$$\exp : \mathbb{R} \to (0, \infty)$$
is a bijection (one-to-one correspondence).

Proof. By the previous proposition, we know that the exponential function has image $(0, \infty)$. Additionally, since exp is strictly increasing, it is injective. Hence, exp is bijective. □

Definition (Logarithm)**.** The *(natural) logarithm* is the function

$$\log : (0, \infty) \to \mathbb{R}$$

defined by

$$\log(x) = \exp^{-1}(x).$$

In particular, for all $x \in \mathbb{R}$ and $y > 0$, we have

$$\log(\exp(x)) = x, \quad \text{and} \quad \exp(\log(y)) = y.$$

Theorem (Inverse Function Theorem (1D))**.** Let $A, B \subseteq \mathbb{R}$, and suppose $f : A \to B$ is a bijection that is differentiable at x_0 with $f'(x_0) \neq 0$. Then f^{-1} is differentiable at $f(x_0)$, and

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Theorem (Properties of log)**.**

(a) For all $x \in (0, \infty)$,

$$\log'(x) = \frac{1}{x}.$$

Hence, for all $0 < a < b$,

$$\int_a^b \frac{1}{x}\, dx = \log(b) - \log(a).$$

(b) For all $x, y \in (0, \infty)$,

$$\log(xy) = \log x + \log y.$$

(c) $\log(1) = 0$ and for all $x > 0$,

$$\log\left(\frac{1}{x}\right) = -\log(x).$$

(d) For all $x > 0$ and $y \in \mathbb{R}$,

$$\log(x^y) = y \log(x).$$

(e) For all $x \in (-1, 1)$,

$$\log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

In particular,

$$\log(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x - 1)^n$$

on $(0, 2)$, so $\log(x)$ is analytic at $x = 1$.

Fact. The function $\log(x)$ is analytic on all of $x \in (0, \infty)$.

Proof. The function $\frac{1}{x}$ is analytic on $(0, \infty)$ as a function from $(0, \infty) \to \mathbb{R}$. Thus, integrating it will yield a power series representation for $\log(x)$. □

Definition. (Alternative definition of exponentiation) Let $a > 0$ and $b \in \mathbb{R}$. We can define

$$a^b = \exp(b \cdot \log(a)).$$

By continuity, this agrees with the usual definition. If $r_n \to b$, then

$$a^{r_n} = \exp(r_n \cdot \log(a))$$

and by continuity,

$$\exp(b \cdot \log(a)) = a^b.$$

Theorem. (Properties of Sine and Cosine)

(a) For all $x \in \mathbb{R}$,

$$\sin^2(x) + \cos^2(x) = 1.$$

In particular,

$$\sin(x) \in [-1, 1], \quad \cos(x) \in [-1, 1].$$

(b) The derivatives satisfy

$$\sin'(x) = \cos(x), \quad \cos'(x) = -\sin(x).$$

(c) The parity properties hold:

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x).$$

(d) The addition formulas:

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$
$$\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x).$$

(e) At zero,

$$\sin(0) = 0, \quad \cos(0) = 1.$$

Definition. $\pi = \inf \{x \in (0, \infty) \mid \sin(x) = 0\}$.

Remark. Since $\sin'(c) = \cos(c) = 1$, we have $\sin(x) > 0$ for $x \in (0, \epsilon)$. So, $\pi > 0$. That is, π is a nonzero number.

Theorem (Theorem 4.7.5 (Periodicity of Trigonometric Functions))**.** Let x be a real number.

(a) We have

$$\cos(x + \pi) = -\cos(x) \quad \text{and} \quad \sin(x + \pi) = -\sin(x).$$

In particular,

$$\cos(x + 2\pi) = \cos(x), \quad \sin(x + 2\pi) = \sin(x),$$

i.e., sin and cos are periodic with period 2π.

(b) We have $\sin(x) = 0$ if and only if x/π is an integer.

(c) We have $\cos(x) = 0$ if and only if x/π is an integer plus $1/2$.

Example. Consider the function

$$f(x) := \sum_{n=0}^{\infty} \varphi^n \cos(3^n \pi x).$$

This function is continuous on $\mathbb{R}$ but differentiable nowhere.

This is an example of the **Weierstrass function**.

Fact. If f and g are analytic on (a, b) and suppose f, g agree on some subinterval (c, d), then $f = g$ on (a, b). That is, analytic functions are determined by their behavior in some small interval. This is a kind of uniqueness result.

$\Rightarrow$ kinda like infinite polynomial interpolation.

Dense Sets

Definition. Let (X, d) be a metric space and let $E \subseteq X$. We say that E is **dense** in X if its closure is all of X, i.e.,

$$\overline{E} = X.$$

Fact. If E is dense, then for every non-empty open set $U \subseteq X$, we have

$$U \cap E \neq \emptyset.$$

Example. Consider $(\mathbb{R}, d_{\text{std}})$:

- $\mathbb{Q}$ is dense.
- $\mathbb{R} \setminus \mathbb{Q}$ is also dense. (Can always find some rational/irrational arbitrarily close.)

Consider (X, d_{disc}):

- E is dense if and only if $E = X$.

Recall

Definition. A set A is called **countable** if there exists an injective function $f : \mathbb{N} \to A$.

Example.

- $\mathbb{Z}$, $\mathbb{Q}$ are countable sets.
- $\mathbb{R}$ is uncountable. Hence, so is $\mathbb{R} \setminus \mathbb{Q}$.

Fact.

- If A_n is a countable sequence of sets, then

$$\bigcup_{n=1}^{\infty} A_n$$

  is countable.

- If A, B are countable, then so is $A \times B$.